

EXISTENCE OF SMOOTH SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN THREE-DIMENSIONAL EUCLIDEAN SPACE

GENQIAN LIU

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China. E-mail address: liugqz@bit.edu.cn

ABSTRACT. Based on the essential connection of the parabolic inertial Lamé equations and Navier-Stokes equations, we prove the existence of smooth solutions of the incompressible Navier-Stokes equations in three-dimensional Euclidean space $\mathbb{R}^3$ by showing the existence and uniqueness of smooth solutions of the parabolic inertial Lamé equations and by letting one Lamé constant λ tend to infinity (the other Lamé constant $\mu > 0$ is fixed).

1. Introduction

The Navier-Stokes equations describe the motion of a viscous fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), \dots, u_n(\mathbf{x}, t)) \in \mathbb{R}^n$ and pressure $p(\mathbf{x}, t) \in \mathbb{R}$, defined for position $\mathbf{x} \in \mathbb{R}^n$ and time $t \geq 0$. The incompressible Navier-Stokes equations are then given by

$$(1.1) \quad \frac{\partial u_j}{\partial t} - \mu \Delta u_j + \frac{\partial p}{\partial x_j} + \sum_{k=1}^n u_k \frac{\partial u_j}{\partial x_k} = f_j(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \quad t \geq 0, \quad j = 1, \dots, n,$$

$$(1.2) \quad \operatorname{div} \mathbf{u} = \sum_{k=1}^n \frac{\partial u_k}{\partial x_k} = 0, \quad \mathbf{x} \in \mathbb{R}^n, \quad t \geq 0$$

with initial conditions

$$(1.3) \quad \mathbf{u}(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

Here, $\boldsymbol{\phi}(\mathbf{x})$ is a given, C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_j(\mathbf{x}, t)$ are the components of a given, externally applied force, μ is a positive constant (the viscosity coefficient), and $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ is the Laplacian in the space variables. Equation (1.1) is just Newton's law $m\mathbf{a} = \mathbf{f}$ for a fluid element subject to the external force $\mathbf{f} = (f_1(\mathbf{x}, t), \dots, f_n(\mathbf{x}, t))$ and to the forces arising from pressure and friction. Equation (1.2) just says that the fluid is incompressible. For physically reasonable solutions, we will restrict attention to forces $\mathbf{f}$ and initial conditions $\boldsymbol{\phi}$ that satisfy

$$(1.4) \quad |\partial_{\mathbf{x}}^\alpha \boldsymbol{\phi}(\mathbf{x})| \leq C_{\alpha M} (1 + |\mathbf{x}|)^{-M} \quad \text{on } \mathbb{R}^n, \quad \text{for any } \alpha \text{ and } M$$

and

$$(1.5) \quad |\partial_t^l \partial_{\mathbf{x}}^\alpha \mathbf{f}(\mathbf{x}, t)| \leq C_{l\alpha M} (1 + |\mathbf{x}| + t)^{-M} \quad \text{on } \mathbb{R}^n \times [0, \infty), \quad \text{for any } l, \alpha, M,$$

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where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index (i.e., an n -tuple of nonnegative integers). A solution of (1.1), (1.2), (1.3) is said to be physically reasonable (see, [12]) only if it satisfies

$$(1.6) \quad p, \mathbf{u} \in \mathbf{C}^\infty(\mathbb{R}^n \times [0, \infty))$$

and

$$(1.7) \quad \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 dx < C \quad \text{for all } t \geq 0 \quad (\text{bounded energy}),$$

where $dx = dx_1 \cdots dx_n$.

It is a fundamental problem in analysis to decide *whether such smooth, physically reasonable solutions exist for the Navier-Stokes equations* (see, statement (A) of [12] by C. L. Fefferman).

Many important progresses and works have been made for the above problem in the past decades. In two dimensions, the existence and uniqueness of smooth solutions of the Navier-Stokes equations has been known for a long time (see, Leray [24], Ladyzhenskaya [23], Sohr [40] or Temam [47]). The existence of the global smooth solution in three dimension is proved if the initial velocity ϕ satisfies a smallness condition. Strong solutions in the three-dimensional case are known to exist for a short interval of time whose length depends on the physical data of the initial value problem (see, Theorem 3.4 on p.104 of [34]). Moreover, this strong solution is known to be unique and to depend continuously on the initial data (see, for example, [7, 23, 34, 47]). Leray in [24] showed that the Navier-Stokes equations (1.1) in three space dimensions always have a weak solution $(\mathbf{u}, p)$ with suitable growth properties, but the uniqueness of weak solutions of the Navier-Stokes equations is not known. Under additional assumptions of more integrability of $\mathbf{u}$, Serrin [39] proved existence of smooth solutions. In [8], Constantin-Fefferman showed smoothness of solutions with a constraint on vorticity. Based on the work of Scheffer [38], Caffarelli-Kohn-Nirenberg in [5] gave a partial regularity result of the Navier-Stokes equations that the dimension of the set of singular points is at most one. Later Lin [26] simplified the proofs of the results in Caffarelli-Kohn-Nirenberg [5]. We refer the reader to the paper [12] of Fefferman, the survey [46] of Temam and the books of Majda-Bertozzi [34] and Lions [28, 29] for more details and references therein.

In this paper, we will solve this long-standing open problem. We state our main result as the following, which is corresponding to the statement (A) in [12].

Theorem 1.1. *Take $\mu > 0$ and $n = 3$. Let $\phi(\mathbf{x})$ be any smooth, divergence-free vector field satisfying (1.4). Take $\mathbf{f}(\mathbf{x}, t)$ to be identically zero. Then there exist smooth functions $u_j(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ on $\mathbb{R}^n \times [0, \infty)$ that satisfy (1.1), (1.2), (1.3), (1.6), (1.7). Moreover, the solution $(\mathbf{u}, p)$ is unique.*

The main ideas and methods of this paper are as follows. We first introduce the following initial value problem of the parabolic inertial Lamé equations:

$$(1.8) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

where μ and λ are the Lamé constants satisfying

$$(1.9) \quad \mu > 0, \lambda + \mu \geq 0.$$

In physics, (1.8) is a fundamental nonlinear system of elastodynamics, which describes the deformation (motion) of an (inertial) elasticity material (body) in $\mathbb{R}^n$ (see, Chapter 5 [25], or Chapter 6 of [35]). Because a lot of powerful mathematical tools (techniques) can be applied to such nonlinear parabolic equations rather than directly to the Navier-Stokes equations, (1.8) will play a key role in this paper. Note that in the above equations, $(\mathbf{u} \cdot \nabla) \mathbf{u}$ is said to be the inertial term. Particularly, when $\lambda + \mu = 0$, (1.8) reduces to the famous n -dimensional viscous Burgers equations

(see, [18], [48], [37], or p.324 of [45]). Our first task is to prove that for the given λ, μ satisfying (1.9) and initial value ϕ satisfying (1.4), there exists a unique smooth solution $\mathbf{u}_{\lambda, \mu}(\mathbf{x}, t)$ in $C([0, \infty); \mathbf{H}^{m+1}(\mathbb{R}^3)) \cap C^\infty(\mathbb{R}^3 \times [0, \infty))$ for the initial value problem of parabolic inertial Lamé equations (1.8). The second task is to show that for any fixed $T > 0$ and divergence-free initial value ϕ and for any given sequence $-\mu \leq \lambda_1 < \lambda_2 < \dots < \lambda_m < \dots \rightarrow +\infty$, there is a subsequence $\{\lambda_{m_l}\}_{l=1}^{+\infty}$ such that the solutions $\mathbf{u}_{\lambda_{m_l}, \mu}(\mathbf{x}, t)$ of (1.8) weakly converge to $\mathbf{u}_\mu(\mathbf{x}, t)$ in $L^2([0, T]; \mathbf{H}^1(\mathbb{R}^3))$ as $l \rightarrow +\infty$ (of course, the functions $-(\lambda_{m_l} + \mu) \operatorname{div} \mathbf{u}_{\lambda_{m_l}, \mu}(\mathbf{x}, t)$ weakly converge to a function $p_\mu(\mathbf{x}, t)$ in $L^2([0, T]; \mathbf{L}^2(\mathbb{R}^3))$ as $l \rightarrow \infty$), as well as $\frac{\partial}{\partial t} \mathbf{u}_{\lambda_{m_l}, \mu}(\mathbf{x}, t)$ weakly converges to $\frac{\partial}{\partial t} \mathbf{u}_\mu(\mathbf{x}, t)$ in $L^2([0, T]; \mathbf{L}^2(\mathbb{R}^3))$ as $l \rightarrow \infty$. In addition, we can immediately show $\operatorname{div} \mathbf{u}_\mu = 0$ in $\mathbb{R}^3 \times [0, T]$. Because $T > 0$ is an arbitrary real number, $(\mathbf{u}_\mu(t, \mathbf{x}), p_\mu(t, \mathbf{x}))$ is just a global smooth strong solution of the Navier-Stokes equations with the viscosity $\mu > 0$ and the same initial value ϕ , and (1.7) holds. The above idea stems from the works [30], [31], [32], [33] and [43] of the present author for studying inverse problems and eigenvalue problems for the Stokes equations, Lamé equations and Navier-Stokes equations, in which the author solved an open problem for determining the viscosity function of the Navier-Stokes equations and discovered the deep relationship between the Lamé eigenvalues and Stokes eigenvalues. These results inspired the author to ultimately prove the global existence of smooth solutions of the Navier-Stokes equations first by showing the existence and uniqueness of global smooth solutions of the parabolic inertial Lamé equations and then by considering the essential (limit) relationship of the above two systems of equations.

In order to prove the existence, smooth and uniqueness of the solutions for the initial value problem (1.8) of the (nonlinear) parabolic inertial Lamé equations in $\mathbb{R}^3$, we will use the following approach: Set $L_{\lambda, \mu} := -\mu \Delta - (\lambda + \mu) \operatorname{grad} \operatorname{div}$ and $\Phi(\mathbf{u}) := (\mathbf{u} \cdot \nabla) \mathbf{u}$. By converting (1.8) to the integral equation

$$(1.10) \quad \mathbf{u}(t) = e^{-tL_{\lambda, \mu}} \phi + \int_0^t e^{-(t-s)L_{\lambda, \mu}} \Phi(\mathbf{u}(s)) ds := \Psi \mathbf{u}(t),$$

and by applying the contraction mapping principle to Ψ (where $\Psi \mathbf{u}(t) := e^{-tL_{\lambda, \mu}} \phi + \int_0^t e^{-(t-s)L_{\lambda, \mu}} \Phi(\mathbf{u}(s)) ds$), we first show the existence of the solution $\mathbf{u}_{\lambda, \mu}$ of (1.8) in a short time interval $[0, T]$ (which is just the unique fixed point in $C([0, T]; \mathbf{H}^m(\mathbb{R}^3))$), and then we prove this solution is also smooth. We prove a key inequality of the fundamental solution and show $\|\mathbf{u}_{\lambda, \mu}\|_{C(\mathbb{R}^3 \times [0, T])} \leq C \|\phi\|_{C_0(\mathbb{R}^3)}$ (similar to the parabolic sup norm estimates, i.e., the maximum principle) as well as $\|\mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{H}^1(\mathbb{R}^3)}^2 \leq \|\mathbf{u}_{\lambda, \mu}(0)\|_{\mathbf{H}^1(\mathbb{R}^3)}^2 e^{\int_0^t c_1 \|\mathbf{u}_{\lambda, \mu}(s)\|_{C(\mathbb{R}^3)}^2 ds} \leq \|\phi\|_{\mathbf{H}^1(\mathbb{R}^3)}^2 e^{tC c_1 \|\phi\|_{C(\mathbb{R}^3)}^2}$, where the constants c_1 and C is independent of λ , and depends only on μ . This rule out the blow up of $\mathbf{u}_{\lambda, \mu}$ in $C([0, T]; \mathbf{H}^m(\mathbb{R}^3))$ for any fixed $T > 0$, and then the local solution can be extended to whole time interval $[0, \infty)$. Employing bootstrap argument and standard regularity technique, we therefore establish the global existence of smooth solution for the parabolic inertial Lamé equations in $\mathbb{R}^3 \times [0, \infty)$. Of course, the solution is also unique. It is a surprising fact that for fixed $\mu > 0$ and divergence-free initial ϕ satisfying (1.4), $\{\mathbf{u}_{\lambda, \mu}\}_{\lambda \in [-\mu, +\infty)}$ and $\{\nabla \mathbf{u}_{\lambda, \mu}\}_{\lambda \in [-\mu, +\infty)}$ are bounded sets of $L^2([0, T]; \mathbf{L}^2(\mathbb{R}^3))$, and $\{\frac{\partial}{\partial t} \mathbf{u}_{\lambda, \mu}(t)\}_{\lambda \in [-\mu, \infty)}$ is also a bounded set of $L^2([0, T]; \mathbf{L}^2(\mathbb{R}^3))$. So the weak limit $\mathbf{u}_\mu(\mathbf{x}, t)$ of smooth solutions $\mathbf{u}_{\lambda_{m_l}, \mu}(\mathbf{x}, t)$ of the parabolic inertial Lamé equations is exactly the smooth solutions of the Navier-Stokes equations. Let us point out that through compatibility conditions for parabolic equations, initial data in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ (respectively, divergence-free Schwartz space) ensure that the solution of the parabolic inertial Lamé equations (respectively, the Navier-Stokes equations) can be smoothly extended to $\mathbb{R}^3 \times [0, \infty)$, including C^∞ -continuity at the initial time $t = 0$.

With a completely similar method, we can prove the following:

Theorem 1.2. Take $\mu > 0$ and $n = 3$. Let ϕ be any smooth, divergence-free vector field in $\mathbb{R}^3$ satisfying

$$(1.11) \quad \phi(\mathbf{x} + \mathbf{e}_j) = \phi(\mathbf{x}) \quad \text{for } 1 \leq j \leq 3,$$

where $\mathbf{e}_j = j^{\text{th}}$ unit vector in $\mathbb{R}^3$. Then there exist smooth functions $p(\mathbf{x}, t)$, $u_j(\mathbf{x}, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1.1), (1.2), (1.6) with $\mathbf{u}(\mathbf{x}, 0) = \phi(\mathbf{x})$, $\mathbf{f} \equiv 0$ and

$$(1.12) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + \mathbf{e}_j, t), \quad p(\mathbf{x}, t) = p(\mathbf{x} + \mathbf{e}_j, t) \quad \text{on } \mathbb{R}^3 \times [0, \infty) \quad \text{for } 1 \leq j \leq 3.$$

This paper is organized as follows. In Section 2, we give the explicit representations of solutions for the linear parabolic Lamé equations and the Stokes equations. In Section 3, we explicitly construct the fundamental solution and establish its basic inequalities for the general parabolic Lamé (elasticity) equations, and then estimate the bound of $\|e^{-tL_{\lambda,\mu}}\|_{\mathcal{L}(\mathbf{H}^m(\mathbb{R}^n), \mathbf{H}^{m+1}(\mathbb{R}^n))}$. The existence, uniqueness, smoothness in $(\mathbf{x}, t)$ and uniform norm boundedness with respect to λ for the solutions $\mathbf{u}_{\lambda,\mu}$ of the parabolic inertial Lamé equations are proved in Section 4. Finally, in Section 5 we prove the main theorem 1.1 by letting $\lambda \rightarrow +\infty$.

2. Explicit representations of solutions for the linear parabolic Lamé equations and Stokes equations

In this section, we will consider explicit representations of solutions for the following initial value problems of linear parabolic Lamé (elasticity) equations

$$(2.1) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{v}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n \end{cases}$$

and the Stokes equations

$$(2.2) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} + \operatorname{grad} p = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{v}(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Throughout this paper, we always assume that (Lamé coefficient) μ is a fixed positive constant and $\lambda + \mu \geq 0$. When $\lambda + \mu = 0$, (2.1) reduces to the initial value problem of the classical heat equations. Clearly, the initial value problem (2.1) can be rewritten as

$$(2.3) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} + L_{\lambda,\mu} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{v}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

where the Lamé operator $L_{\lambda,\mu}$ is given by

$$(2.4) \quad L_{\lambda,\mu} \mathbf{w} = \left\{ -\mu \left(\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \right) \mathbf{I}_n - (\mu + \lambda) \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{bmatrix} \right\} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Furthermore, we have

$$(2.5) \quad L_{\lambda,\mu} \mathbf{w}(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} (\iota(L_{\lambda,\mu})) \begin{pmatrix} \hat{w}_1(\boldsymbol{\xi}) \\ \vdots \\ \hat{w}_n(\boldsymbol{\xi}) \end{pmatrix} d\boldsymbol{\xi},$$

where

$$(2.6) \quad \iota(L_{\lambda,\mu}) = \mu|\boldsymbol{\xi}|^2 \mathbf{I}_n + (\mu + \lambda) \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix}$$

is the full symbol of the operator $L_{\lambda,\mu}$, $\hat{w}_j$ is the Fourier transform of w_j defined by $\hat{w}_j(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}} w_j(\mathbf{x}) d\mathbf{x}$, $|\boldsymbol{\xi}|^2 = \sum_{k=1}^n \xi_k^2$ and $d\boldsymbol{\xi} = d\xi_1 \cdots d\xi_n$. Let us denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing (vector-valued) functions:

$$(2.7) \quad \mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \mid |\partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})| \leq C_{\alpha M} (1 + |\mathbf{x}|)^{-M} \text{ on } \mathbb{R}^n \text{ for any } \alpha \text{ and } M\},$$

and by $\mathbf{C}_0^m(\mathbb{R}^n)$ the space of C^m -smooth (vector-valued) functions vanishing at infinity: $\mathbf{C}_0^m(\mathbb{R}^n) = \{\phi \in \mathbf{C}^m(\mathbb{R}^n) \mid |\partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow +\infty, 0 \leq |\alpha| \leq m\}$. We denote by $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ the volume of the unit ball in $\mathbb{R}^n$. The convolution $f(\mathbf{x}) * g(\mathbf{x})$ of two functions on $\mathbb{R}^n$ is defined by $f(\mathbf{x}) * g(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mathbf{y}$ provided that the integrals in question exist. Then we have the following:

Theorem 2.1. *Let the Lamé constants μ and λ satisfy $\mu > 0$, $\lambda + \mu \geq 0$, and let $n \geq 3$.*

(a) *If $\mathbf{v}_{\lambda,\mu}$ is the unique solution of the linear parabolic Lamé equations (2.1) with $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\mathbf{v}_{\lambda,\mu}$ can be explicitly represented as*

$$(2.8) \quad \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * \psi(\mathbf{x}),$$

where $\psi(\mathbf{x}) := \frac{1}{n(n-2)\omega_n|\mathbf{x}|^{n-2}} * (\text{grad div } \phi(\mathbf{x})) \in \mathbf{C}_0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ is the unique solution of Poisson's equations

$$(2.9) \quad \begin{cases} -\Delta \psi = \text{grad div } \phi & \text{in } \mathbb{R}^n, \\ \lim_{|\mathbf{x}| \rightarrow +\infty} \psi(\mathbf{x}) = 0. \end{cases}$$

Moreover,

$$(2.10) \quad \mathbf{v}_{\lambda,\mu} \in \mathbf{C}^\infty(\mathbb{R}^n \times [0, \infty)), \quad \int_{\mathbb{R}^n} |\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)|^2 d\mathbf{x} < C \text{ for all } t \geq 0.$$

(b) *If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\text{div } \phi = 0$ in $\mathbb{R}^n$, then the solution $\mathbf{v}_\mu$ of the Stokes equations (2.2) can be explicitly represented as*

$$(2.11) \quad \mathbf{v}_\mu(\mathbf{x}, t) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x})$$

and $p(t, \mathbf{x}) \equiv c(t)$ in $\mathbb{R}^n \times [0, \infty)$.

Proof. (a) Since the Lamé operator $L_{\lambda,\mu}$ is uniformly strongly elliptic (see, for example, (10.4) on p. 297 of [36], Chapters 5–6 of [19], (5.150) of [25], [1], [6], [35]), it follows from Chap. 6 of [35], Stewart [42], Browder [4], Friedman [13] that $-L_{\lambda,\mu}$ generates analytic semigroups $(e^{-zL_{\lambda,\mu}})_{z \in \Lambda}$ on $\mathbf{C}_0(\mathbb{R}^n)$, on $\mathbf{L}^2(\mathbb{R}^n)$, and on $\mathbf{L}^p(\mathbb{R}^n)$, where Λ is a sector in the complex plane $\mathbb{C}$ (cf. [42]). The corresponding strongly continuous semigroups $(e^{-tL_{\lambda,\mu}})_{t \geq 0}$ can be obtained from the resolvent operator $(\vartheta \mathbf{I} - L_{\lambda,\mu})^{-1}$ of $L_{\lambda,\mu}$ by the Cauchy integral formula (see, Chapter I of [10] or p. 366 of [17]):

$$(2.12) \quad e^{-tL_{\lambda,\mu}} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\vartheta} (\vartheta \mathbf{I} - L_{\lambda,\mu})^{-1} d\vartheta, \quad t \geq 0,$$

where $\mathcal{C}$ is a suitable curve (contour) in the complex plane in the positive direction around the positive real axis, and $\mathbf{I}$ is the identity operator. Therefore, we have

$$(2.13) \quad \begin{aligned} e^{-tL_{\lambda,\mu}}\phi(\mathbf{x}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \left(\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\vartheta} \iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}) d\vartheta \right) \hat{\phi}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} \left(\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\vartheta} \iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}) d\vartheta \right) \phi(\mathbf{y}) d\mathbf{y} d\boldsymbol{\xi}, \end{aligned}$$

where $\iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1})$ is the full symbol of the resolvent operator $(\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}$. If we denote

$$(2.14) \quad \mathbf{Z}_{\lambda,\mu}(\mathbf{x}, t) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \left(\frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\vartheta} \iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}) d\vartheta \right) d\boldsymbol{\xi},$$

then

$$(2.15) \quad e^{-tL_{\lambda,\mu}}\phi(\mathbf{x}) = \mathbf{Z}_{\lambda,\mu}(\mathbf{x}, t) * \phi(\mathbf{x}) = \int_{\mathbb{R}^n} \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t) \phi(\mathbf{y}) d\mathbf{y}$$

by (2.13).

Next, we will explicitly calculate $\iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1})$. By a direct calculation we find that

$$(2.16) \quad \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} = |\boldsymbol{\xi}|^2 \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix}.$$

Thus, the following two matrices play a key role:

$$F := \left\{ \mathbf{I}_n, \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} \right\}.$$

The set F of above two matrices can generate a matrix ring $\mathfrak{F}$ according to the usual matrix addition and multiplication of $\mathfrak{F}$ on the ring $\mathfrak{N}$ of functions. This implies that $\iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1})$ must have the following form:

$$(2.17) \quad \iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}) = s_1 \mathbf{I}_n + s_2 \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix},$$

where s_1 and s_2 are unknown functions which will be determined later. This key idea is inspired by Galois group theory for solving the polynomial equation (see [3] or [9]). Substituting (2.17) into $\iota(\vartheta\mathbf{I} - L_{\lambda,\mu}) \iota((\vartheta\mathbf{I} - L_{\lambda,\mu})^{-1}) = \mathbf{I}_n$, we have

$$(2.18) \quad \left\{ (\vartheta - \mu|\boldsymbol{\xi}|^2) \mathbf{I}_n - (\mu + \lambda) \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} \right\} \left\{ s_1 \mathbf{I}_n + s_2 \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} \right\} = \mathbf{I}_n,$$

i.e.,

$$s_1 (\vartheta - \mu|\boldsymbol{\xi}|^2) \mathbf{I}_n + \left\{ s_2 (\vartheta - \mu|\boldsymbol{\xi}|^2) - s_1(\mu + \lambda) - s_2(\mu + \lambda)|\boldsymbol{\xi}|^2 \right\} \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix} = \mathbf{I}_n.$$

Since the set F is a basis of the matrix ring $\mathfrak{F}$, we get

$$\begin{cases} s_1 (\vartheta - \mu|\boldsymbol{\xi}|^2) = 1, \\ s_2 (\vartheta - \mu|\boldsymbol{\xi}|^2) - s_1(\mu + \lambda) - s_2(\mu + \lambda)|\boldsymbol{\xi}|^2 = 0. \end{cases}$$

It follows that

$$(2.19) \quad \begin{cases} s_1 = \frac{1}{\vartheta - \mu|\boldsymbol{\xi}|^2}, \\ s_2 = \frac{\mu + \lambda}{(\vartheta - \mu|\boldsymbol{\xi}|^2)(\vartheta - (2\mu + \lambda)|\boldsymbol{\xi}|^2)}. \end{cases}$$

Therefore

$$(2.20) \quad \iota((\vartheta \mathbf{I} - L_{\lambda, \mu})^{-1}) = \frac{1}{\vartheta - \mu|\boldsymbol{\xi}|^2} \mathbf{I}_n + \frac{\mu + \lambda}{(\vartheta - \mu|\boldsymbol{\xi}|^2)(\vartheta - (2\mu + \lambda)|\boldsymbol{\xi}|^2)} \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix}.$$

Now, by applying the residue theorem (see, for example, Chap. 4, §5 in [2]) we have

$$(2.21) \quad \begin{aligned} \frac{1}{2\pi i} \int_C e^{-t\vartheta} \iota((\vartheta \mathbf{I} - L_{\lambda, \mu})^{-1}) d\vartheta &= e^{-t\mu|\boldsymbol{\xi}|^2} \mathbf{I}_n \\ &+ \frac{1}{|\boldsymbol{\xi}|^2} \left(e^{-t(\lambda+2\mu)|\boldsymbol{\xi}|^2} - e^{-t\mu|\boldsymbol{\xi}|^2} \right) \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix}. \end{aligned}$$

Noting that

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} e^{-t\mu|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}}, \quad \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} e^{-t(\lambda+2\mu)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} = \frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}}$$

and (see, p. 194, (2) of Ch. II in [14])

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \frac{1}{|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} &= \frac{1}{(2\pi)^n} \left(2^{n-2} \pi^{\frac{n}{2}} \Gamma\left(\frac{n-2}{2}\right) |\mathbf{x}|^{2-n} \right) \\ &= \frac{1}{n(n-2)\omega_n} |\mathbf{x}|^{2-n}, \quad n > 2. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \left(\frac{1}{|\boldsymbol{\xi}|^2} e^{-t\mu|\boldsymbol{\xi}|^2} \right) d\boldsymbol{\xi} &= \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \left(\frac{1}{n(n-2)\omega_n} |\mathbf{x}|^{2-n} \right) \\ &= \int_{\mathbb{R}^n} \frac{1}{n(n-2)\omega_n |\mathbf{z}|^{n-2}} \frac{e^{-|\mathbf{x}-\mathbf{z}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} dz, \\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \left(\frac{1}{|\boldsymbol{\xi}|^2} e^{-t(\lambda+2\mu)|\boldsymbol{\xi}|^2} \right) d\boldsymbol{\xi} &= \frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} * \left(\frac{1}{n(n-2)\omega_n} |\mathbf{x}|^{2-n} \right) \\ &= \int_{\mathbb{R}^n} \frac{1}{n(n-2)\omega_n |\mathbf{z}|^{n-2}} \frac{e^{-|\mathbf{x}-\mathbf{z}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} dz. \end{aligned}$$

Inserting these and (2.21) into (2.14) we obtain

$$(2.22) \quad \mathbf{Z}_{\lambda, \mu}(\mathbf{x}, t) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \mathbf{I}_n + \begin{bmatrix} \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x}, t)}{\partial x_1^2} & \cdots & \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x}, t)}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x}, t)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x}, t)}{\partial x_n^2} \end{bmatrix},$$

where

$$(2.23) \quad \begin{aligned} W_{\lambda,\mu}(\mathbf{x}, t) &:= \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * \frac{1}{n(n-2)\omega_n|\mathbf{x}|^{n-2}} \\ &= \int_{\mathbb{R}^n} \frac{1}{n(n-2)\omega_n|\mathbf{z}|^{n-2}} \left(\frac{e^{-|\mathbf{x}-\mathbf{z}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}-\mathbf{z}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) d\mathbf{z}. \end{aligned}$$

Clearly, the function $W_{\lambda,\mu}(\mathbf{x}, t)$ is well-defined and smooth for all $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, because the integral in (2.23) is absolutely convergent and the convolution $W_{\lambda,\mu}(\mathbf{x}, t)$ has the same smoothness as $\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}}$ (cf. p.17 or p.100 of [20]), and $\int_{\mathbb{R}^n} W_{\lambda,\mu}(\mathbf{x}, t) d\mathbf{x} = 0$. In particular, for any fixed $t > 0$ and any multi-index α we have $\partial_{\mathbf{x}}^{\alpha} W_{\lambda,\mu}(\cdot, t) \in C_0(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$. It follows from (2.15) that for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, the action of strongly continuous semigroups $(e^{-tL_{\lambda,\mu}})_{t \geq 0}$ on ϕ can be further represented as

$$(2.24) \quad \begin{aligned} e^{-tL_{\lambda,\mu}} \phi(\mathbf{x}) &= \mathbf{Z}_{\lambda,\mu}(\mathbf{x}, t) * \phi(\mathbf{x}) \\ &= \int_{\mathbb{R}^n} \frac{e^{-|\mathbf{y}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \phi(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^n} \begin{bmatrix} \frac{\partial^2 W_{\lambda,\mu}(\mathbf{y}, t)}{\partial y_1^2} & \dots & \frac{\partial^2 W_{\lambda,\mu}(\mathbf{y}, t)}{\partial y_1 \partial y_n} \\ \vdots & & \vdots \\ \frac{\partial^2 W_{\lambda,\mu}(\mathbf{y}, t)}{\partial y_n \partial y_1} & \dots & \frac{\partial^2 W_{\lambda,\mu}(\mathbf{y}, t)}{\partial y_n^2} \end{bmatrix} \phi(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \frac{e^{-|\mathbf{y}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \phi(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^n} W_{\lambda,\mu}(\mathbf{y}, t) (\text{grad div } \phi(\mathbf{x} - \mathbf{y})) d\mathbf{y} \\ &= \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) + W_{\lambda,\mu}(\mathbf{x}, t) * (\text{grad div } \phi(\mathbf{x})) \\ &= \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * \left(\frac{1}{n(n-2)\omega_n|\mathbf{x}|^{n-2}} * (\text{grad div } \phi(\mathbf{x})) \right) \\ &= \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * \psi(\mathbf{x}), \end{aligned}$$

where we integrated by parts twice for the third equality, and used the convolution properties $f(\mathbf{x}) * g(\mathbf{x}) = g(\mathbf{x}) * f(\mathbf{x})$ and $(f(\mathbf{x}) * g(\mathbf{x})) * h(\mathbf{x}) = f(\mathbf{x}) * (g(\mathbf{x}) * h(\mathbf{x}))$ for the last equality. Note that $\psi(\mathbf{x}) := \frac{1}{n(n-2)\omega_n|\mathbf{x}|^{n-2}} * (\text{grad div } \phi(\mathbf{x}))$ is just the Newtonian potential of $(\text{grad div } \phi(\mathbf{x}))$, i.e., ψ is the unique solution of Poisson's equations

$$\begin{cases} -\Delta \psi = \text{grad div } \phi & \text{in } \mathbb{R}^n, \\ \psi(\mathbf{x}) \rightarrow 0 & \text{as } |\mathbf{x}| \rightarrow +\infty. \end{cases}$$

It follows from regularity of elliptic equations that $\partial_{\mathbf{x}}^{\alpha} \psi \in C_0(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$ for given $\phi \in \mathcal{S}(\mathbb{R}^n)$ and any multi-index α .

According to the properties of strongly continuous semigroups on $C_0(\mathbb{R}^n)$ (see, Theorem 6 of [42]), we get that for $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t) := e^{-tL_{\lambda,\mu}} \phi(\mathbf{x})$ satisfies the linear parabolic Lamé equations:

$$(2.25) \quad \begin{cases} \frac{\partial \mathbf{v}_{\lambda,\mu}}{\partial t} - \mu \Delta \mathbf{v}_{\lambda,\mu} - (\lambda + \mu) \text{grad div } \mathbf{v}_{\lambda,\mu} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{v}_{\lambda,\mu}(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

and the solution $\mathbf{v}_{\lambda,\mu}$ is unique among vector fields with $\mathbf{v}_{\lambda,\mu}(\cdot, t) \in \mathbf{C}_0(\mathbb{R}^n)$, $t \geq 0$. Moreover, $\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)$ is continuous in $t \geq 0$ uniformly in $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)$ is continuously differentiable in $t > 0$ uniformly in $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{v}_{\lambda,\mu}(\cdot, t) \in \mathbf{C}^1(\mathbb{R}^n) \cap \mathbf{W}_{loc}^{2,q}(\mathbb{R}^n)$ with $q > n$ for any fixed $t > 0$ (see, [42]). In addition, for the given initial value $\phi \in \mathcal{S}(\mathbb{R}^n)$, we see that $(L_{\lambda,\mu})^m \phi \in \mathbf{C}_0(\mathbb{R}^n) \cap \mathbf{C}^\infty(\mathbb{R}^n)$ for all $m \geq 0$, that is, any order compatibility conditions hold (see, Theorem 6 on p.388 of [11]) for initial problem (2.25). It is easy to verify by the regularity of solutions for linear parabolic system (see, for example, Theorem 7 on p.390 of [11], or [13]) that the solution $\mathbf{v}_{\lambda,\mu} = e^{-tL_{\lambda,\mu}}\phi$ of the linear parabolic Lamé equations (2.1) belongs to $\mathbf{C}^\infty(\mathbb{R}^n \times [0, \infty))$. On the other hand, since $L_{\lambda,\mu}$ also generates a strongly continuous semigroup on $\mathbf{L}^2(\mathbb{R}^n)$, we get $\mathbf{v}_{\lambda,\mu}(\cdot, t) \in \mathbf{L}^2(\mathbb{R}^n)$ for each $t \geq 0$. By applying (2.25) and Green's formula (see, [22], [33] or [32]), we have

$$\begin{aligned} \frac{1}{2} \frac{\partial \|\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)}^2}{\partial t} &= \int_{\mathbb{R}^n} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t) \cdot \left(\mu \Delta \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t) + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t) \right) dx \\ &= - \int_{\mathbb{R}^n} \left(2\mu (\operatorname{Def} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t), \operatorname{Def} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)) + \lambda (\operatorname{div} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t))^2 \right) dx \\ &= - \int_{\mathbb{R}^n} \left(\mu |\nabla \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)|^2 \right) dx < 0, \end{aligned}$$

where $\operatorname{Def} \mathbf{v} := \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ and $(\operatorname{Def} \mathbf{v}, \operatorname{Def} \mathbf{v}) := \frac{1}{4} \sum_{j,k=1}^n \left| \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right|^2$. This implies that $\int_{\mathbb{R}^n} |\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)|^2 dx$ is a decreasing function of t in $[0, +\infty)$, and hence

$$\int_{\mathbb{R}^n} |\mathbf{v}_{\lambda,\mu}(\mathbf{x}, t)|^2 dx \leq \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^2 dx < C \text{ for all } t \geq 0.$$

(b) For $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{div} \phi = 0$ in $\mathbb{R}^n$, we set

$$\mathbf{v}_\mu(\mathbf{x}, t) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{e^{-|\mathbf{y}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \phi(\mathbf{x} - \mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, t > 0.$$

Then $\operatorname{div}_{\mathbf{x}}(\mathbf{v}_\mu(\mathbf{x}, t)) = 0$ in $\mathbb{R}^n \times [0, \infty)$, and $\mathbf{v}_\mu(\mathbf{x}, t)$ is the unique solution of the equations

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t > 0, \\ \operatorname{div} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t > 0, \\ \mathbf{v}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

This implies that $\mathbf{v}_\mu(\mathbf{x}, t)$ is the solution of the Stokes equations (2.2) with $p(\mathbf{x}, t) \equiv c(t)$. $\square$

Remark 2.2. *i) When $\lambda + \mu = 0$, our explicit representation (2.8) for the solutions of (2.1) reduces to the classical representation of the heat equations (see, for example, [21] or [13]).*

ii) The explicit representation (2.8) (as well as the fundamental solution (2.22)) for the solutions of initial problem of linear parabolic Lamé equations will play an important role later.

iii) Throughout this paper, we always choose initial-value ϕ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The reason is that this kind of initial-value can ensure the solution of equations being C^∞ -smoothly extended to $\mathbb{R}^n \times [0, \infty)$ (including C^∞ -smoothly extended to initial time $t = 0$) according to the parabolic compatibility conditions (cf. p. 288 and p. 290 of [11]).

3. Construction and inequalities of fundamental solutions for general linear parabolic Lamé equations

3.1. Construction of fundamental solutions. In order to discuss the solutions $\mathbf{u}$ of (1.8), we need to give explicit representations of solutions for the following general linear inhomogeneous

parabolic Lamé equations:

$$(3.1) \quad \begin{cases} \left(\frac{\partial}{\partial t} - \mu \Delta - (\lambda + \mu) \operatorname{grad} \operatorname{div} + (\mathbf{b} \cdot \nabla) + \mathbf{c} \right) \mathbf{v} = \mathbf{f}, & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ \mathbf{v}(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n \end{cases}$$

with $\boldsymbol{\phi} \in \mathcal{S}(\mathbb{R}^n)$, where the coefficient $\mathbf{b}(\mathbf{x}, t) = (b_1(\mathbf{x}, t), \dots, b_n(\mathbf{x}, t))$ is a smooth vector-valued function, $\mathbf{c}(\mathbf{x}, t) = (c_{jk}(\mathbf{x}, t))_{n \times n}$ is a smooth matrix-valued function, and $\mathbf{b}(\mathbf{x}, t)$, $\mathbf{c}(\mathbf{x}, t)$ are assumed to be bounded and smooth. We denote $M := -\mu \Delta - (\lambda + \mu) \operatorname{grad} \operatorname{div} + \mathbf{b} \cdot \nabla + \mathbf{c}$. Sometimes, we will write $\frac{\partial^l}{\partial t^l}$, $\frac{\partial^\alpha}{\partial x^\alpha}$, $\frac{\partial^l}{\partial t^l} \frac{\partial^\alpha}{\partial x^\alpha}$, $\dots$ as ∂_t^l , ∂_x^α , $\partial_t^l \partial_x^\alpha$, $\dots$, respectively. In Chapter 9 of [13], for more general parabolic system (i.e., (1.6) on p.239 of [13] with $f_j = 0$)

$$(3.2) \quad \frac{\partial v_j}{\partial t} = \sum_{l=1}^N \sum_{|\alpha| \leq 2p} A_{\alpha}^{jl}(\mathbf{x}, t) \partial_x^\alpha v_l, \quad j = 1, \dots, N,$$

its fundamental solution has been constructed by the parametrix method (of E. E. Levi). An $N \times N$ matrix $\mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau)$ of functions defined for $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$, $(\mathbf{y}, \tau) \in \mathbb{R}^n \times [0, T]$, $t > \tau$ is said to be a fundamental solution (or a fundamental matrix) of (3.2) if, as a function of $(\mathbf{x}, t)$ ($\mathbf{x} \in \mathbb{R}^n$, $\tau < t \leq T$), satisfies (3.2) (i.e., each column is a solution of (3.2)), and is such that

$$(3.3) \quad \lim_{t \rightarrow \tau^+} \int_{\mathbb{R}^n} \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) \boldsymbol{\phi}(\mathbf{y}) d\mathbf{y} = \boldsymbol{\phi}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

for any continuous function $\boldsymbol{\phi}(\mathbf{y})$ in $\mathbb{R}^n$ (see, p.240 of [13]).

Now, we will construct the explicit fundamental solution $\mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau)$ for the linear parabolic Lamé system (3.1) with $\mathbf{f} \equiv 0$ according to such a method. Rewrite the system (3.1), for $\mathbf{f} \equiv 0$, in the form

$$(3.4) \quad \frac{\partial \mathbf{v}}{\partial t} = -M \mathbf{v} = (-L_{\lambda, \mu} - (\mathbf{b} \cdot \nabla) - \mathbf{c}) \mathbf{v},$$

where $L_{\lambda, \mu} = -\mu \Delta \mathbf{v} - (\lambda + \mu) \operatorname{grad} \operatorname{div}$ is the principal part of M , which is defined in (2.4) of Section 2. Let $\mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)$ be the fundamental solution of the system

$$(3.5) \quad \frac{\partial \mathbf{v}}{\partial t} = -L_{\lambda, \mu} \mathbf{v} \quad \text{in } \mathbb{R}^n \times [0, \infty)$$

constructed in Section 2 (here the coefficients of $L_{\lambda, \mu}$ is taken at the fixed point $(\mathbf{w}, t)$ with $\mathbf{w} \in \mathbb{R}^n$, $t > \tau$). Because $L_{\lambda, \mu}$ is a uniformly strongly elliptic (vector-valued) operator with constant coefficients (cf. [1]), the “first approximation” $L_{\lambda, \mu}$ to M is independent of spatial parameter $\mathbf{w} \in \mathbb{R}^n$, so that we may take $\mathbf{w} = \mathbf{y}$, and $\mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)$ (see, p.248 of [13]) can be exactly written as (cf. Section 2):

$$(3.6) \quad \mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) = \frac{e^{-|\mathbf{x} - \mathbf{y}|^2 / 4\mu(t - \tau)}}{(4\pi\mu(t - \tau))^{n/2}} \mathbf{I}_n + \begin{bmatrix} \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau)}{\partial x_1^2} & \dots & \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau)}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau)}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 W_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau)}{\partial x_n^2} \end{bmatrix},$$

where

$$(3.7) \quad W_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau) := \left(\frac{e^{-|\mathbf{x} - \mathbf{y}|^2 / 4(\lambda + 2\mu)(t - \tau)}}{(4\pi(\lambda + 2\mu)(t - \tau))^{n/2}} - \frac{e^{-|\mathbf{x} - \mathbf{y}|^2 / 4\mu(t - \tau)}}{(4\pi\mu(t - \tau))^{n/2}} \right) * \frac{1}{n(n - 2)\omega_n |\mathbf{x} - \mathbf{y}|^{n-2}} \\ = \int_{\mathbb{R}^n} \frac{1}{n(n - 2)\omega_n |z|^{n-2}} \left(\frac{e^{-|\mathbf{x} - \mathbf{y} - z|^2 / 4(\lambda + 2\mu)(t - \tau)}}{(4\pi(\lambda + 2\mu)(t - \tau))^{n/2}} - \frac{e^{-|\mathbf{x} - \mathbf{y} - z|^2 / 4\mu(t - \tau)}}{(4\pi\mu(t - \tau))^{n/2}} \right) dz.$$

Note that for the fundamental solution $\mathbf{Z}_{\lambda, \mu}(\mathbf{x}, t)$ in Section 2, we have taken $\tau \equiv 0$ and replaced $\mathbf{x} - \mathbf{y}$ by $\mathbf{x}$; but in this section, we always take $0 \leq \tau < t \leq T$ as in [13] and simply write $\mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t - \tau)$

as $\mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)$. Following the parameter method (see, p. 252, proof of Theorem 2 in §4 of Chapter 9 of [13]), we have

$$(3.8) \quad \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) = \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) + \int_{\tau}^t d\sigma \int_{\mathbb{R}^n} \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{w}, t; \mathbf{w}, \sigma) \mathbf{\Phi}(\mathbf{w}, \sigma; \mathbf{y}, \tau) d\mathbf{w}$$

with $\mathbf{\Phi}$ is an $n \times n$ matrix. Let $\mathbf{K}$ be the $n \times n$ matrix whose j th row K_j is given by

$$(3.9) \quad \begin{aligned} K_j(\mathbf{x}, t; \mathbf{y}, \tau) &= [-M_j(\mathbf{x}, t, \partial_{\mathbf{x}}) - \delta_j \partial_t] \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) \\ &= [-M_{0j}(\mathbf{x}, t, \partial_{\mathbf{x}}) + M_{0j}(\mathbf{y}, t, \partial_{\mathbf{x}})] \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) - M_{1j}(\mathbf{x}, t, \partial_{\mathbf{x}}) \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) \\ &= ((-\mathbf{b}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}}) - \mathbf{c}(\mathbf{x}, t)) \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)_j, \end{aligned}$$

where $\delta_j \partial_t \mathbf{Z}_{\lambda,\mu} = \partial_t Z_{\lambda,\mu}^j$, $M_{0j}(\mathbf{x}, t, \partial_{\mathbf{x}}) = (L_{\lambda,\mu})_j$ and $M_{1j}(\mathbf{x}, t, \partial_{\mathbf{x}}) = (\mathbf{b}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} + \mathbf{c}(\mathbf{x}, t))_j$. Here we have used the fact that $M_{0j}(\mathbf{x}, t, \partial_{\mathbf{x}}) = M_{0j}(\mathbf{y}, t, \partial_{\mathbf{x}}) = (L_{\lambda,\mu})_j$ for all $(\mathbf{x}, t), (\mathbf{y}, t) \in \mathbb{R}^n \times [0, \infty)$. It follows from the proof of Theorem 2 of Chpter 9, §2 of [13] that the series

$$(3.10) \quad \mathbf{\Phi}(\mathbf{x}, t; \mathbf{y}, \tau) = \sum_{m=1}^{\infty} \mathbf{K}_m(\mathbf{x}, t; \mathbf{y}, \tau)$$

is convergent, where $\mathbf{K}_1 = \mathbf{K}$,

$$(3.11) \quad \mathbf{K}_m(\mathbf{x}, t; \mathbf{y}, \tau) = \int_{\tau}^t d\sigma \int_{\mathbb{R}^n} \mathbf{K}_1(\mathbf{x}, t; \mathbf{w}, \sigma) \mathbf{K}_{m-1}(\mathbf{w}, \sigma; \mathbf{y}, \tau) d\mathbf{w},$$

and

$$(3.12) \quad \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) = \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) + \int_{\tau}^t d\sigma \int_{\mathbb{R}^n} \mathbf{Z}_{\lambda,\mu}(\mathbf{x} - \mathbf{w}, t; \mathbf{w}, \sigma) \sum_{m=1}^{\infty} \mathbf{K}_m(\mathbf{w}, \sigma; \mathbf{y}, \tau) d\mathbf{w}$$

is a fundamental solution of (3.4). The proof for the convergence of the above series (3.12) can be found in §4 of Chap. 9 in [13].

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$, and let $\mathbf{f}(\mathbf{x}, t)$ be a continuous vector-valued function in $\mathbb{R}^n \times [0, T]$, Hölder continuous in $\mathbf{x}$ uniformly in bounded subset of $\mathbb{R}^n \times [0, T]$, and $|\mathbf{f}(\mathbf{x}, t)| \leq A e^{a|\mathbf{x}|^2}$ in $\mathbb{R}^n \times [0, T]$. It follows from (4.27) on p. 256–257 of [13] and Theorem 3 of [13] that

$$(3.13) \quad \mathbf{v}(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, 0) \phi(\mathbf{y}) d\mathbf{y} - \int_0^t \int_{\mathbb{R}^n} \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) \mathbf{f}(\mathbf{y}, \tau) d\tau d\mathbf{y}$$

is a solution of the initial value problem (3.1).

3.2. Basic inequalities. For any integers $r, l \geq 0$, let us set

$$(3.14) \quad \begin{aligned} \|\phi\|_{C^r(\mathbb{R}^n)} &:= \max_{|\alpha| \leq r; \mathbf{x} \in \mathbb{R}^n} |\partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x})|, \\ \|\mathbf{v}\|_{C^{r,l}(\mathbb{R}^n \times [0, t])} &= \max_{|\alpha| \leq r; k \leq l; \mathbf{x} \in \mathbb{R}^n; 0 \leq \sigma \leq t} |\partial_{\sigma}^k \partial_{\mathbf{x}}^{\alpha} \mathbf{v}(\mathbf{x}, \sigma)|, \\ \|\mathbf{E}\| &:= \max_{1 \leq j, k \leq n} |E_{jk}| \text{ for a matrix } \mathbf{E} = (E_{jk})_{n \times n}. \end{aligned}$$

On p. 251 and p. 255 of [13], the following inequality was proved by A. Friedman for the fundamental solution $\mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau)$ of (3.1) with $\mathbf{f} \equiv \mathbf{0}$:

$$(3.15) \quad \|\partial_{\mathbf{x}}^{\alpha} \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau)\| \leq \frac{C_{\alpha}}{(t - \tau)^{(n+|\alpha|)/2}} e^{-c_{\alpha} |\mathbf{x} - \mathbf{y}|^2 / (t - \tau)},$$

where $0 \leq |\alpha| \leq 2$, and C_{α} and c_{α} are positive constants depending only on λ, μ , on bounds of $|\partial_{\mathbf{x}}^h \mathbf{b}(\mathbf{x}, t)|$ and $|\partial_{\mathbf{x}}^h \mathbf{c}(\mathbf{x}, t)|$, ($|h| \leq |\alpha|$). However, the inequalities (3.15) are not enough to obtain the

global existence and regularity of solutions for the parabolic inertial Lamé system and Navier-Stokes equations. This forces us to look for stronger inequalities.

It is well-known that for the initial value problem of scalar parabolic equation

$$(3.16) \quad \begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + \sum_{j=1}^n b_j(\mathbf{x}, t) \frac{\partial u}{\partial x_j} + c(t, \mathbf{x})u = 0 & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

with continuous and bounded functions $b_j(\mathbf{x}, t)$ and $c(\mathbf{x}, t) \geq 0$, the maximum principle holds, from which one has the following sup norm estimates:

$$(3.17) \quad \|u(\mathbf{x}, t)\|_{C(\mathbb{R}^n \times [0, T])} \leq \|\phi\|_{C(\mathbb{R}^n)}.$$

In other words, the corresponding parabolic operator semigroup $T_t \phi(\mathbf{x}) := u(\mathbf{x}, t)$ of (3.16) is contraction on $C_0(\mathbb{R}^n)$, and (3.17) can also be got by the inequality of the fundamental solution:

$$(3.18) \quad \|\Gamma(\mathbf{x}, t; \mathbf{y}, \tau)\| \leq \frac{e^{-|\mathbf{x}-\mathbf{y}|^2/4\mu(t-\tau)}}{(4\pi\mu(t-\tau))^{n/2}} \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } t > \tau,$$

where $\Gamma(\mathbf{x}, t; \mathbf{y}, \tau)$ is the fundamental solution of (3.16).

For a general linear parabolic Lamé system, the maximum principle may not hold true. However, when the coefficients of equations are constants, we have the following theorem which plays similar role as the fundamental solution and the sup norm estimates:

Theorem 3.1. *Let $\mu > 0$ and $\lambda + \mu \geq 0$. Assume that $\mathbf{b}$ is a real constant vector and c is a non-negative constant. Suppose that $\mathbf{u}(\mathbf{x}, t)$ (respectively, $\Gamma(\mathbf{x}, t; \mathbf{y}, \tau)$) is the solution of initial problem (respectively, fundamental solution) of linear parabolic Lamé system*

$$(3.19) \quad \begin{cases} \left(\frac{\partial}{\partial t} - \mu \Delta - (\lambda + \mu) \operatorname{grad} \operatorname{div} + (\mathbf{b} \cdot \nabla) + c \right) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, t > \tau \geq 0, \\ \mathbf{u}(\mathbf{x}, \tau) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

(i) *Then the following inequality holds*

$$(3.20) \quad \|\Gamma(\mathbf{x}, t; \mathbf{y}, \tau)\| \leq \frac{C_1}{(t-\tau)^{n/2}} e^{-c_1 |\mathbf{x}-\mathbf{y}|^2/(t-\tau)},$$

furthermore,

$$(3.21) \quad \|\mathbf{u}(\mathbf{x}, t)\|_{C(\mathbb{R}^n \times [0, T])} \leq C \|\phi\|_{C(\mathbb{R}^n)},$$

where the positive constants C_1 , c_1 and C are independent of λ , $\mathbf{b}$, $c \geq 0$ and T , and depend only on the Lamé constant $\mu > 0$.

(ii) *If $\mathbf{b} = 0$ and $c = 0$, then for any $|\alpha| \geq 0$, the following inequality holds*

$$(3.22) \quad \|\partial_{\mathbf{x}}^{\alpha} \Gamma(\mathbf{x}, t; \mathbf{y}, \tau)\| = \|\partial_{\mathbf{x}}^{\alpha} \mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)\| \leq \frac{C_{\alpha}}{(t-\tau)^{(n+|\alpha|)/2}} e^{-c_{\alpha} |\mathbf{x}-\mathbf{y}|^2/(t-\tau)},$$

where the constants C_{α} , c_{α} are independent of λ and T , and depend only on the Lamé constant $\mu > 0$.

Lemma 3.2 (see, Lemma 1 on p.242 of [13]). *Let $g(\zeta)$ be a continuous real-valued function in the n -dimensional complex space C^n and assume that $g(\kappa\zeta) = \kappa^2 g(\zeta)$ for any $\kappa \geq 0$, $\zeta \in C^n$, and*

$$(3.23) \quad g(\xi) \leq -\sigma |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n,$$

where σ is some positive constant. Then there exists a constant $a > 0$ depending only on σ such that

$$(3.24) \quad g(\xi + i\eta) \leq -\frac{\sigma}{2} |\xi|^2 + a |\eta|^2 \text{ for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n.$$

Proof of Theorem 3.1. Let $\mathbf{v}(\mathbf{x}, t; \tau)$ be the (fundamental) solution of

$$(3.25) \quad \begin{cases} \left(\frac{\partial}{\partial t} - \mu \Delta - (\lambda + \mu) \operatorname{grad} \operatorname{div} + (\mathbf{b} \cdot \nabla) + c \right) \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^n, t > \tau \geq 0, \\ \mathbf{v}(\mathbf{x}, \tau; \tau) = \delta(\mathbf{x}) \mathbf{I}_n, & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

We write tentatively

$$(3.26) \quad \mathbf{v}(\mathbf{x}, t; \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi},$$

where $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$ is the Fourier transform of $\mathbf{v}(\mathbf{x}, t; \tau)$ with respect to $\mathbf{x}$. Clearly,

$$\partial_{\mathbf{x}}^{\alpha} \mathbf{v}(\mathbf{x}, t; \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} (i\boldsymbol{\xi})^{\alpha} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi}.$$

Substituting (3.26) into (3.25), we have

$$(3.27) \quad \begin{cases} \frac{1}{(2\pi)^n} \left[\frac{\partial}{\partial t} \left(\int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} \right) - \mu \Delta \left(\int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} \right) - (\lambda + \mu) \operatorname{grad} \operatorname{div} \left(\int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} \right) \right. \\ \left. + (\mathbf{b} \cdot \nabla) \left(\int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} \right) + c \left(\int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} \right) \right] = 0, & \mathbf{x} \in \mathbb{R}^n, t > \tau \geq 0, \\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} (\hat{\mathbf{v}}(\boldsymbol{\xi}, \tau; \tau) - \mathbf{I}_n) d\boldsymbol{\xi} = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

i.e.,

$$(3.28) \quad \begin{cases} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \left[\frac{\partial}{\partial t} + \mu |\boldsymbol{\xi}|^2 \mathbf{I}_n + (\lambda + \mu) \boldsymbol{\xi} \boldsymbol{\xi}^T + i(\mathbf{b} \cdot \boldsymbol{\xi}) \mathbf{I}_n + c \mathbf{I}_n \right] \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi} = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^n, t > \tau, \\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} (\hat{\mathbf{v}}(\boldsymbol{\xi}, \tau; \tau) - \mathbf{I}_n) d\boldsymbol{\xi} = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

where $\boldsymbol{\xi} \boldsymbol{\xi}^T := \begin{bmatrix} \xi_1^2 & \cdots & \xi_1 \xi_n \\ \vdots & & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n^2 \end{bmatrix}$. We first calculate the solution $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$ of the symbol equations

$$(3.29) \quad \begin{cases} \frac{d\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)}{dt} = -\mathbf{A} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau), & \mathbf{x} \in \mathbb{R}^n, t > \tau, \\ \hat{\mathbf{v}}(\boldsymbol{\xi}, \tau; \tau) = \mathbf{I}_n, & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

where $\mathbf{A} := \mu |\boldsymbol{\xi}|^2 \mathbf{I}_n + (\lambda + \mu) \boldsymbol{\xi} \boldsymbol{\xi}^T + i(\mathbf{b} \cdot \boldsymbol{\xi}) \mathbf{I}_n + c \mathbf{I}_n$. Set

$$\mathbf{B} := \int_{\tau}^t \left[\mu |\boldsymbol{\xi}|^2 \mathbf{I}_n + (\lambda + \mu) \boldsymbol{\xi} \boldsymbol{\xi}^T + i(\mathbf{b} \cdot \boldsymbol{\xi}) \mathbf{I}_n + c \mathbf{I}_n \right] ds.$$

For any $n \times n$ matrix $\mathbf{M}$, the matrix exponential $e^{\mathbf{M}}$ is defined by Taylor series $e^{\mathbf{M}} := \sum_{k=0}^{\infty} \frac{\mathbf{M}^k}{k!}$. It is well-known that the solution $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$ of the ordinary differential system (3.29) can be obtained from the resolvent expression $(\vartheta \mathbf{I}_n - \mathbf{B})^{-1}$ of matrix-valued function $\mathbf{B}$ by the Cauchy integral formula:

$$e^{-\mathbf{B}} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\vartheta} (\vartheta \mathbf{I}_n - \mathbf{B})^{-1} d\vartheta, \quad t > \tau,$$

where $\vartheta \in \mathbb{C}$, and $\mathcal{C}$ is a suitable curve (contour) in the complex plane $\mathbb{C}$ in the positive direction around the spectrums of $\mathbf{B}$. Therefore (see, for example, p. 16, §4 of Chapt. 1 of [44]),

$$(3.30) \quad \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) = e^{-(t-\tau)(\mu |\boldsymbol{\xi}|^2 \mathbf{I}_n + (\lambda + \mu) \boldsymbol{\xi} \boldsymbol{\xi}^T) - \int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi}) \mathbf{I}_n + c \mathbf{I}_n] ds}.$$

The above representation (3.30) follows from the matrix calculus of (matrix-valued) exponential function:

$$\frac{d}{dt}e^{-\mathbf{B}} = e^{-\mathbf{B}} \left(-\frac{d\mathbf{B}}{dt} \right) = \left(-\frac{d\mathbf{B}}{dt} \right) e^{-\mathbf{B}} = -\mathbf{A}e^{-\mathbf{B}},$$

and the fact that the three matrices $-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)$, $-\int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n] ds$, $-\int_{\tau}^t [c\mathbf{I}_n] ds$ are commutable each other. Since $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$ is the unique solution of (3.29), it follows from (3.28) that

$$\mathbf{v}(\mathbf{x}, t; \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi}$$

is the solution of (3.25). $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$ is also called the Green matrix of (3.29) (see, p. 241 of [13]).

Now, we will estimate $\|\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)\|$ (i.e., sup norm of the Green matrix $\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)$). Clearly,

$$(3.31) \quad \begin{aligned} & e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T) - \int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n + c\mathbf{I}_n] ds} \\ &= e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)} e^{-\int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n] ds} e^{-\int_{\tau}^t [c\mathbf{I}_n] ds}. \end{aligned}$$

Here we have used the well-known result that $e^{\mathbf{E}_1 + \mathbf{E}_2} = e^{\mathbf{E}_1} e^{\mathbf{E}_2}$ holds for any two $n \times n$ matrices $\mathbf{E}_1$ and $\mathbf{E}_2$ if and only if $\mathbf{E}_1 \mathbf{E}_2 = \mathbf{E}_2 \mathbf{E}_1$. Since $i(\mathbf{b} \cdot \boldsymbol{\xi})$ is a purely imaginary-valued function and $c \geq 0$, we have $\|e^{-\int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n] ds}\| \leq 1$ and $\|e^{-\int_{\tau}^t [c\mathbf{I}_n] ds}\| \leq 1$, which lead to

$$(3.32) \quad \|e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T) - \int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n + c\mathbf{I}_n] ds}\| \leq \|e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)}\|.$$

It remains to estimate $\|e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)}\|$. Applying the Cauchy integral formula, we see (cf. (2.12)) that

$$e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-(t-\tau)\vartheta} [\vartheta\mathbf{I}_n - \mu|\boldsymbol{\xi}|^2 - (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T]^{-1} d\vartheta.$$

In Section 2, we have given the explicit expression: $(\vartheta\mathbf{I}_n - \mu|\boldsymbol{\xi}|^2 - (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1} = \iota((\vartheta\mathbf{I} - L_{\lambda, \mu})^{-1}) = \frac{1}{\vartheta - \mu|\boldsymbol{\xi}|^2} \mathbf{I}_n + \frac{\mu + \lambda}{(\vartheta - \mu|\boldsymbol{\xi}|^2)(\vartheta - (2\mu + \lambda)|\boldsymbol{\xi}|^2)} \boldsymbol{\xi}\boldsymbol{\xi}^T$ (see, (2.18) and (2.20)). Combining this and (2.21) we have

$$(3.33) \quad e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)} = e^{-(t-\tau)\mu|\boldsymbol{\xi}|^2} \mathbf{I}_n + \frac{1}{|\boldsymbol{\xi}|^2} (e^{-(t-\tau)(\lambda+2\mu)|\boldsymbol{\xi}|^2} - e^{-(t-\tau)\mu|\boldsymbol{\xi}|^2}) \boldsymbol{\xi}\boldsymbol{\xi}^T.$$

Noting that $\lambda + 2\mu \geq \mu$ and $\frac{|\xi_j \xi_k|}{|\boldsymbol{\xi}|^2} \leq \frac{1}{2}$ for all $1 \leq j, k \leq n$, we find by (3.33) that

$$(3.34) \quad \|e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2 + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T)}\| \leq 2e^{-(t-\tau)\mu|\boldsymbol{\xi}|^2} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

Combining (3.30), (3.31), (3.32) and (3.34), we obtain

$$(3.35) \quad \|\hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau)\| = \|e^{-(t-\tau)(\mu|\boldsymbol{\xi}|^2\mathbf{I}_n + (\lambda+\mu)\boldsymbol{\xi}\boldsymbol{\xi}^T) - \int_{\tau}^t [i(\mathbf{b} \cdot \boldsymbol{\xi})\mathbf{I}_n] ds} e^{-\int_{\tau}^t [c\mathbf{I}_n] ds}\| \leq 2e^{-(t-\tau)\mu|\boldsymbol{\xi}|^2}.$$

Set

$$g(\boldsymbol{\zeta}) = -\mu|\boldsymbol{\zeta}|^2 \quad \text{for all } \boldsymbol{\zeta} = \boldsymbol{\xi} + i\boldsymbol{\eta} \in \mathbb{C}^n.$$

Clearly, $g(\kappa\boldsymbol{\zeta}) = -\kappa^2\mu|\boldsymbol{\zeta}|^2$ for any $\kappa \geq 0$, $\boldsymbol{\zeta} \in \mathbb{C}^n$, and

$$g(\boldsymbol{\xi}) = -\mu|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

It follows from Lemma 3.2 that there exists a constant $a > 0$ depending only on the Lamé constant μ such that

$$(3.36) \quad g(\boldsymbol{\xi} + i\boldsymbol{\eta}) \leq -\frac{\mu}{2}|\boldsymbol{\xi}|^2 + a|\boldsymbol{\eta}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{\eta} \in \mathbb{R}^n.$$

Thus, in view of (3.35) and (3.36) we obtain

$$(3.37) \quad |||\hat{\mathbf{v}}(\boldsymbol{\xi} + i\boldsymbol{\eta}, t; \tau)||| \leq 2e^{(t-\tau)\left(-\frac{\mu}{2}|\boldsymbol{\xi}|^2 + a|\boldsymbol{\eta}|^2\right)}.$$

On the other hand, we will regard (3.29) as complex ordinary differential equations (i.e., complexification of variables $\boldsymbol{\xi}$ of the given differential equations). By a standard theorem for ordinary differential equations, it follows that $\hat{\mathbf{v}}(\boldsymbol{\zeta}, t; \tau)$ are also entire functions of $\boldsymbol{\zeta} := \boldsymbol{\xi} + i\boldsymbol{\eta}$. Applying Cauchy's (integral) theorem and (3.37), we find that the matrix-valued function

$$(3.38) \quad \boldsymbol{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) = \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y}) \cdot (\boldsymbol{\xi} + i\boldsymbol{\eta})} \hat{\mathbf{v}}(\boldsymbol{\xi} + i\boldsymbol{\eta}, t; \tau) d\boldsymbol{\xi} \quad (t > \tau)$$

is independent of $\boldsymbol{\eta}$ and, hence, coincides with the matrix-valued function defined by

$$(3.39) \quad \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) = \mathbf{v}(\mathbf{x}, t; \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} \hat{\mathbf{v}}(\boldsymbol{\xi}, t; \tau) d\boldsymbol{\xi}.$$

This point of view stems from a discussion in lines 8–11 on p. 245 of [13]. Using (3.37) and (3.38), we get

$$(3.40) \quad |||\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)||| \leq \frac{2}{(2\pi)^n} e^{-\boldsymbol{\eta} \cdot (\mathbf{x} - \mathbf{y})} e^{a(t-\tau)|\boldsymbol{\eta}|^2} \int_{\mathbb{R}^n} e^{-\frac{\mu}{2}(t-\tau)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi}.$$

Noting that for any $p > 0$ we can choose $q := p$ such that

$$(3.41) \quad pq = (p^2 + q^2)/2.$$

Taking $\eta_k = |\eta_k| \operatorname{sgn}(x_k - y_k)$, $|\eta_1| = \dots = |\eta_n|$,

$$p = \frac{|\mathbf{x} - \mathbf{y}|}{n[2a(t-\tau)]^{1/2}},$$

and then choosing $|\boldsymbol{\eta}|$ such that (3.41) holds for $q = [2a(t-\tau)]^{1/2}|\boldsymbol{\eta}|$, we obtain

$$-\boldsymbol{\eta} \cdot (\mathbf{x} - \mathbf{y}) + (t-\tau)a|\boldsymbol{\eta}|^2 \leq -|\boldsymbol{\eta}| \frac{|\mathbf{x} - \mathbf{y}|}{n} + (t-\tau)a|\boldsymbol{\eta}|^2 = -\frac{a_1|\mathbf{x} - \mathbf{y}|^2}{t-\tau}$$

for some positive constant $a_1 := 1/(4n^2a)$ depending only on μ . Substituting this into (3.40) we get

$$(3.42) \quad |||\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)||| \leq \frac{2}{(2\pi)^n} e^{-a_1|\mathbf{x}-\mathbf{y}|^2/(t-\tau)} \int_{\mathbb{R}^n} e^{-\frac{\mu}{2}(t-\tau)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi}.$$

Just as the proof from line -5 to line -1 on p. 245 of [13], taking polar coordinates and then substituting $\rho = |\boldsymbol{\xi}|(t-\tau)^{1/2}$ we find that the integral on the right-hand side of (3.42) is bounded by a constant times

$$(t-\tau)^{n/2} \int_0^\infty \rho^{n-1} e^{-\frac{\mu}{2}\rho^2} d\rho \leq \text{const. } (t-\tau)^{-n/2}.$$

Substituting this into (3.42) we obtain (3.20), in which the constants C_1 and c_1 are independent of λ , $\mathbf{b}$ and $c \geq 0$, and depend only on the Lamé constant μ .

Moreover, for any $\mathbf{x} \in \mathbb{R}^n$ and $t \in (0, T]$, we have

$$\begin{aligned} |\mathbf{u}(\mathbf{x}, t)| &= \left| \int_{\mathbb{R}^n} \boldsymbol{\Gamma}(\mathbf{x}, t; \mathbf{y}, 0) \boldsymbol{\phi}(\mathbf{y}) d\mathbf{y} \right| \\ &\leq \int_{\mathbb{R}^n} |||\boldsymbol{\Gamma}(\mathbf{x}, t; \mathbf{y}, 0)||| \|\boldsymbol{\phi}\|_{C(\mathbb{R}^n)} d\mathbf{y} \\ &\leq \int_{\mathbb{R}^n} \frac{C_1}{t^{n/2}} e^{-c_1|\mathbf{x}-\mathbf{y}|^2/t} \|\boldsymbol{\phi}\|_{C(\mathbb{R}^n)} d\mathbf{y} = C \|\boldsymbol{\phi}\|_{C(\mathbb{R}^n)}, \end{aligned}$$

which implies the desired result (3.20), where the constant $C = \int_{\mathbb{R}^n} \frac{C_1}{t^{n/2}} e^{-c_1 |\mathbf{x}-\mathbf{y}|^2/t} d\mathbf{y}$ is independent of $\mathbf{b}$, $c \geq 0$ and T , and depends only on μ .

To prove ii) (that is, (3.22) for $|\alpha| \geq 0$ under conditions $\mathbf{b} = 0$ and $c = 0$), as in p. 246 of [13], we use the equality

$$\partial_{\mathbf{x}}^{\alpha} \mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau) = \partial_{\mathbf{x}}^{\alpha} \mathbf{Z}_{\lambda, \mu}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i(\boldsymbol{\xi} + i\boldsymbol{\eta}))^{\alpha} e^{i(\boldsymbol{\xi} + i\boldsymbol{\eta}) \cdot (\mathbf{x} - \mathbf{y})} \hat{\mathbf{v}}(\boldsymbol{\xi} + i\boldsymbol{\eta}, t; \tau) d\boldsymbol{\xi}.$$

It follows this and (3.40) that (cf. p. 246 of [13])

$$(3.43) \quad \begin{aligned} |\partial_{\mathbf{x}}^{\alpha} \mathbf{\Gamma}^{jk}(\mathbf{x} - \mathbf{y}, t; \mathbf{y}, \tau)| &\leq \tilde{c}_1 |\boldsymbol{\eta}|^{|\alpha|} e^{-\boldsymbol{\eta} \cdot (\mathbf{x} - \mathbf{y})} e^{a(t-\tau)|\boldsymbol{\eta}|^2} \int_{\mathbb{R}^n} e^{-\frac{\mu}{2}(t-\tau)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} \\ &\quad + \tilde{c}_1 e^{-\boldsymbol{\eta} \cdot (\mathbf{x} - \mathbf{y})} e^{a(t-\tau)|\boldsymbol{\eta}|^2} \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{|\alpha|} e^{-\frac{\mu}{2}(t-\tau)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} \\ &\equiv J_1 + J_2. \end{aligned}$$

To estimate J_2 we proceed as in the case $\alpha = 0$ and note that

$$\begin{aligned} \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{|\alpha|} e^{-\frac{\mu}{2}(t-\tau)|\boldsymbol{\xi}|^2} d\boldsymbol{\xi} &\leq \text{const.} (t-\tau)^{-(n+|\alpha|)/2} \int_0^{\infty} \rho^{n+|\alpha|-1} e^{-\frac{\mu}{2}\rho^2} d\rho \\ &\quad \text{const.} (t-\tau)^{-(n+|\alpha|)/2}. \end{aligned}$$

Thus J_2 is bounded by the right-hand side of (3.22).

As for J_1 , we also proceed as in p. 246 of [13], with p given, the q for which (3.41) holds is $p = q$. Since $|\boldsymbol{\eta}|$ is chosen so that $q = [2a(t-\tau)]^{1/2} |\boldsymbol{\eta}|$, we get

$$(3.44) \quad |\boldsymbol{\eta}| = a_2 \left(\frac{|\mathbf{x} - \mathbf{y}|}{(t-\tau)^{1/2}} \right) (t-\tau)^{-1/2},$$

where $a_2 := 1/(2na)$ depends only on μ . Replacing the factor $|\boldsymbol{\eta}|^{|\alpha|}$ in J_1 by the $|\alpha|$ th power of the right-hand side of (3.44) and estimating the remaining factors of J_1 as in the case $\alpha = 0$ we see, upon using the inequality $t^{1/2} e^{-\epsilon t} \leq \text{const.}$ for any fixed $\epsilon > 0$ and $0 \leq t < \infty$, that J_1 is also bounded by the right-hand side of (3.22). According to (3.43), the proof of (3.22) for any α is thereby completed. In other words, we get the inequality (3.22) for the fundamental solution $\mathbf{\Gamma}(\mathbf{x}, t; \mathbf{y}, \tau)$. Since all the associated constants in the above discussion depend only on μ and are independent of λ , $\mathbf{b}$, $c \geq 0$ and T , so do the constants C_1 , c_1 , C_{α} , c_{α} . $\square$

Remark 3.3. When $\lambda + \mu = 0$, Theorem 3.1 becomes a classical known result for the fundamental solution of the heat equations (see, for example, [13], [15] and [11]).

Lemma 3.4 (Young's inequality, see, Theorem 1.3 of Chapter I, p. 3 of [41]). If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^n)$ then the convolution $h = g * f$ is well defined and belongs to $L^p(\mathbb{R}^n)$. Moreover,

$$(3.45) \quad \|h\|_{L^p(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Theorem 3.5. Let $n > 2$, and let $\mathbf{H}^m(\mathbb{R}^n) = \{\mathbf{h} \in \mathbf{L}^2(\mathbb{R}^n) \mid \partial^{\alpha} \mathbf{h} \in \mathbf{L}^2(\mathbb{R}^n) \text{ for } |\alpha| \leq m\}$ be the Sobolev space. Suppose that $(e^{-tL_{\lambda, \mu}})_{t \geq 0}$ is the strongly continuous semigroup generated by $-L_{\lambda, \mu}$ on $\mathbf{H}^m(\mathbb{R}^n)$. Then

$$(3.46) \quad \|e^{-tL_{\lambda, \mu}} \phi\|_{\mathbf{H}^{m+1}(\mathbb{R}^n)} \leq Ct^{-1/2} \|\phi\|_{\mathbf{H}^m(\mathbb{R}^n)} \quad \text{for any } \phi \in \mathbf{H}^m(\mathbb{R}^n), \quad 0 < t \leq 1,$$

i.e.,

$$(3.47) \quad \|e^{-tL_{\lambda, \mu}}\|_{\mathcal{L}(\mathbf{H}^m(\mathbb{R}^n), \mathbf{H}^{m+1}(\mathbb{R}^n))} \leq Ct^{-1/2}, \quad \text{for all } 0 < t \leq 1,$$

where the constant C is independent of λ , and depends only on μ .

Proof. From (2.8) of Theorem 2.1 we know

$$(3.48) \quad e^{-tL_{\lambda,\mu}}\phi(\mathbf{x}) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * \phi(\mathbf{x}) + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * \psi(\mathbf{x})$$

where ψ is the Newtonian potential of $\text{grad div } \phi$, i.e., $\psi(\mathbf{x}) = \frac{1}{n(n-2)\omega_n|\mathbf{x}|^{n-2}} * (\text{grad div } \phi(\mathbf{x}))$. Obviously, $\psi := \psi(\phi)$ is a linear vector-valued function of ϕ . Let the multi-indices α, β satisfy $|\alpha| \leq m, |\beta| = 1$. Taking the partial derivatives $\partial_{\mathbf{x}}^{\alpha}$, respectively $\partial_{\mathbf{x}}^{\beta}\partial_{\mathbf{x}}^{\alpha}$ on both sides of (3.48), we get

$$(3.49) \quad \partial_{\mathbf{x}}^{\alpha}(e^{-tL_{\lambda,\mu}}\phi(\mathbf{x})) = \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * (\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})) + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * (\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})),$$

$$(3.50) \quad \partial_{\mathbf{x}}^{\beta}\partial_{\mathbf{x}}^{\alpha}(e^{-tL_{\lambda,\mu}}\phi(\mathbf{x})) = \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})) + \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})).$$

Here we have used the convolution property $\partial^{\alpha}(f * g) = (\partial^{\alpha}f) * g = f * (\partial^{\alpha}g)$. It follows from (3.22) of Theorem 3.1 that there exist positive constants $C_{\beta}, c_{\beta}, \tilde{C}_{\beta}, \tilde{c}_{\beta}$ (which are independent of λ , and depend only μ) such that

$$(3.51) \quad \left| \partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right| \leq \frac{C_{\beta}}{t^{1/2}} \frac{e^{-c_{\beta}|\mathbf{x}|^2/t}}{t^{n/2}}, \quad \left| \partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} \right) \right| \leq \frac{\tilde{C}_{\beta}}{t^{1/2}} \frac{e^{-\tilde{c}_{\beta}|\mathbf{x}|^2/t}}{t^{n/2}}.$$

Combining (3.49)–(3.50) and (3.51), we find by applying Minkowski's inequality and Young's inequality (see, Lemma 3.4) that for any $|\alpha| \leq m$ and $|\beta| = 1$,

$$(3.52) \quad \begin{aligned} \|\partial_{\mathbf{x}}^{\alpha}(e^{-tL_{\lambda,\mu}}\phi(\mathbf{x}))\|_{L^2(\mathbb{R}^n)} &= \left\| \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * (\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})) \right. \\ &\quad \left. + \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * (\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \left\| \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} * (\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\ &\quad + \left\| \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) * (\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \left\| \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right\|_{L^1(\mathbb{R}^n)} \|\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})\|_{L^2(\mathbb{R}^n)} \\ &\quad + \left\| \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right\|_{L^1(\mathbb{R}^n)} \|\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})\|_{L^2(\mathbb{R}^n)} \\ &\leq \|\partial_{\mathbf{x}}^{\alpha}\phi(\mathbf{x})\|_{L^2(\mathbb{R}^n)} + 2 \|\partial_{\mathbf{x}}^{\alpha}\psi(\mathbf{x})\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

$$\begin{aligned}
(3.53) \quad & \left\| \partial_{\mathbf{x}}^{\beta} \partial_{\mathbf{x}}^{\alpha} (e^{-tL_{\lambda, \mu}} \phi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} = \left\| \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x})) \right. \\
& \quad \left. + \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha} \psi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\
& \leq \left\| \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\
& \quad + \left\| \left(\partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right) * (\partial_{\mathbf{x}}^{\alpha} \psi(\mathbf{x})) \right\|_{L^2(\mathbb{R}^n)} \\
& \leq \left\| \partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right\|_{L^1(\mathbb{R}^n)} \left\| \partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)} \\
& \quad + \left\| \partial_{\mathbf{x}}^{\beta} \left(\frac{e^{-|\mathbf{x}|^2/4(\lambda+2\mu)t}}{(4\pi(\lambda+2\mu)t)^{n/2}} - \frac{e^{-|\mathbf{x}|^2/4\mu t}}{(4\pi\mu t)^{n/2}} \right) \right\|_{L^1(\mathbb{R}^n)} \left\| \partial_{\mathbf{x}}^{\alpha} \psi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)} \\
& \leq \left\| \frac{1}{t^{1/2}} \frac{C_{\beta} e^{-c_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} \right\|_{L^1(\mathbb{R}^n)} \left\| \partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)} \\
& \quad + \left(\frac{1}{t^{1/2}} \left\| \frac{\tilde{C}_{\beta} e^{-\tilde{c}_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} \right\|_{L^1(\mathbb{R}^n)} + \frac{1}{t^{1/2}} \left\| \frac{C_{\beta} e^{-c_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} \right\|_{L^1(\mathbb{R}^n)} \right) \left\| \partial_{\mathbf{x}}^{\alpha} \psi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)} \\
& \leq \frac{C'_{\beta}}{t^{1/2}} \left\| \partial_{\mathbf{x}}^{\alpha} \phi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)} + \frac{C'_{\beta} + C''_{\beta}}{t^{1/2}} \left\| \partial_{\mathbf{x}}^{\alpha} \psi(\mathbf{x}) \right\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where the constants $C'_{\beta} = \int_{\mathbb{R}^n} \frac{C_{\beta} e^{-c_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} dx$ and $C''_{\beta} = \int_{\mathbb{R}^n} \frac{\tilde{C}_{\beta} e^{-\tilde{c}_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} dx$ are independent of λ , and depend only on μ . Therefore, for $0 < t \leq 1$,

(3.54)

$$\left\| e^{-tL_{\lambda, \mu}} \phi(\mathbf{x}) \right\|_{\mathbf{H}^{m+1}(\mathbb{R}^n)} \leq \frac{C''_1}{t^{1/2}} \left\| \phi(\mathbf{x}) \right\|_{\mathbf{H}^m(\mathbb{R}^n)} + \frac{C''_1 + C''_2}{t^{1/2}} \left\| \psi(\mathbf{x}) \right\|_{\mathbf{H}^m(\mathbb{R}^n)},$$

where the constants $C''_1 = 1 + \max_{|\beta|=1} \int_{\mathbb{R}^n} \frac{C_{\beta} e^{-c_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} dx$ and $C''_2 = 2 + \max_{|\beta|=1} \int_{\mathbb{R}^n} \frac{\tilde{C}_{\beta} e^{-\tilde{c}_{\beta} |\mathbf{x}|^2/t}}{t^{n/2}} dx$ are independent of λ , and depend only on μ .

For given $\phi \in \mathbf{H}^m(\mathbb{R}^n)$, since $\mathbf{H}^m(\mathbb{R}^n) = \mathbf{H}_0^m(\mathbb{R}^n)$, there exist vector-valued (compact support) functions $\phi_l \in C_c^{\infty}(\mathbb{R}^n)$ such that $\phi_l \rightarrow \phi$ in $\mathbf{H}^m(\mathbb{R}^n)$ as $l \rightarrow +\infty$. We define ψ_l to be the Newtonian potential of $\text{grad div } \phi_l$, i.e.,

$$\begin{cases} -\Delta \psi_l = \text{grad div } \phi_l & \text{in } \mathbb{R}^n, \\ \psi_l(\mathbf{x}) \rightarrow 0 & \text{as } |\mathbf{x}| \rightarrow +\infty. \end{cases}$$

Clearly,

$$\begin{aligned}
(3.55) \quad \psi_l(\mathbf{x}) &= \frac{1}{n(n-2)\omega_n |\mathbf{x}|^{n-2}} * (\text{grad div } \phi_l(\mathbf{x})) \\
&= \text{grad div} \left(\frac{1}{n(n-2)\omega_n |\mathbf{x}|^{n-2}} * \phi_l(\mathbf{x}) \right) := \text{grad div } \tilde{\psi}_l,
\end{aligned}$$

where $\tilde{\psi}_l$ satisfies the Poisson equation

$$(3.56) \quad \begin{cases} -\Delta \tilde{\psi}_l = \phi_l & \text{in } \mathbb{R}^n, \\ \tilde{\psi}_l(\mathbf{x}) \rightarrow 0 & \text{as } |\mathbf{x}| \rightarrow +\infty. \end{cases}$$

It follows from a direct calculation (see also, (3) on p. 328 of [11], or (9.34) on p. 235 of [16]) that

$$\int_{\mathbb{R}^n} |D^2 \tilde{\psi}_l|^2 dx = \int_{\mathbb{R}^n} |\Delta \tilde{\psi}_l|^2 dx,$$

so that by (3.56)

$$\int_{\mathbb{R}^n} |D^2 \tilde{\psi}_l|^2 dx = \int_{\mathbb{R}^n} |\phi_l|^2 dx.$$

This equality and (3.55) imply

$$\begin{aligned} (3.57) \quad \int_{\mathbb{R}^n} |\psi_l|^2 dx &= \int_{\mathbb{R}^n} |\text{grad div } \tilde{\psi}_l|^2 dx \\ &\leq \int_{\mathbb{R}^n} |D^2 \tilde{\psi}_l|^2 dx \\ &= \int_{\mathbb{R}^n} |\phi_l|^2 dx. \end{aligned}$$

From the first equality of (3.55) we have

$$(3.58) \quad \partial_{\mathbf{x}}^{\alpha} \psi_l(\mathbf{x}) = \frac{1}{n(n-2)\omega_n |\mathbf{x}|^{n-2}} * \text{grad div } \partial_{\mathbf{x}}^{\alpha} \phi_l(\mathbf{x}),$$

that is, $\partial_{\mathbf{x}}^{\alpha} \psi_l$ satisfies the following equations

$$\begin{cases} -\Delta(\partial_{\mathbf{x}}^{\alpha} \psi_l) = \text{grad div } \partial_{\mathbf{x}}^{\alpha} \phi_l & \text{in } \mathbb{R}^n, \\ \partial_{\mathbf{x}}^{\alpha} \psi_l(\mathbf{x}) \rightarrow 0 & \text{as } |\mathbf{x}| \rightarrow +\infty. \end{cases}$$

It follows from Green's formula that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} \psi_l)|^2 dx &= \left| - \int_{\mathbb{R}^n} (\partial_{\mathbf{x}}^{\alpha} \psi_l) \cdot (\Delta \partial_{\mathbf{x}}^{\alpha} \psi_l) dx \right| \\ &= \left| \int_{\mathbb{R}^n} (\partial_{\mathbf{x}}^{\alpha} \psi_l) \cdot (\text{grad div } (\partial_{\mathbf{x}}^{\alpha} \phi_l)) dx \right| \\ &= \left| - \int_{\mathbb{R}^n} (\text{div } (\partial_{\mathbf{x}}^{\alpha} \psi_l)) (\text{div } (\partial_{\mathbf{x}}^{\alpha} \phi_l)) dx \right| \\ &\leq \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} \psi_l)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} \phi_l)|^2 dx \right)^{1/2}, \end{aligned}$$

Consequently

$$(3.59) \quad \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} \psi_l)|^2 dx \right)^{1/2} \leq \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} \phi_l)|^2 dx \right)^{1/2} \quad \text{for all } |\alpha| \leq m-1.$$

By a similar way, we can get that

$$(3.60) \quad \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} (\psi_l - \psi_k))|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^n} |\nabla(\partial_{\mathbf{x}}^{\alpha} (\phi_l - \phi_k))|^2 dx \right)^{\frac{1}{2}} \quad \text{for all } 1 \leq l, k < \infty, |\alpha| \leq m-1.$$

Passing to limits in (3.59) and (3.57) as $l \rightarrow \infty$, we deduce

$$(3.61) \quad \|\psi\|_{\mathbf{H}^k(\mathbb{R}^n)} \leq \|\phi\|_{\mathbf{H}^k(\mathbb{R}^n)} \quad \text{for any } 0 \leq k \leq m$$

because of (3.60), where $\psi(\mathbf{x}) = \frac{1}{n(n-2)\omega_n |\mathbf{x}|^{n-2}} * \text{grad div } \phi(\mathbf{x})$. From this and (3.54) we conclude that for any $\phi \in \mathbf{H}^m(\mathbb{R}^n)$,

$$\|e^{-tL_{\lambda, \mu}} \phi(\mathbf{x})\|_{\mathbf{H}^{m+1}(\mathbb{R}^n)} \leq \left(\frac{C_1''}{t^{1/2}} + \frac{C_1'' + C_2''}{t^{1/2}} \right) \|\phi(\mathbf{x})\|_{\mathbf{H}^m(\mathbb{R}^n)}.$$

□

4. Existence and uniqueness of global smooth solutions for the parabolic inertial Lamé equations

In this section we will consider the existence and uniqueness of global smooth solutions of the initial problem for the parabolic inertial Lamé equations with $n = 3$:

$$(4.1) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Lemma 4.1 (Propositions 3.6 and 3.7 of Chapter 13 in [45]). *If $|\beta| + |\gamma| = k$. Then*

$$(4.2) \quad \|(D^\beta f)(D^\gamma g)\|_{L^2} \leq C \|f\|_{L^\infty} \|g\|_{H^k} + C \|f\|_{H^k} \|g\|_{L^\infty},$$

$$(4.3) \quad \|f \cdot g\|_{H^k} \leq C \|f\|_{L^\infty} \|g\|_{H^k} + C \|f\|_{H^k} \|g\|_{L^\infty},$$

for all $f, g \in C_0(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n) = \{\mathbf{f} \in C(\mathbb{R}^n) \mid |\mathbf{f}(\mathbf{x})| \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow +\infty\}$.

Theorem 4.2. *Given $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a $T_0 > 0$ (which is independent of λ) such that the parabolic inertial Lamé equations (4.1) admits a unique solution $\mathbf{u} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n))$, $m > n/2$, with the initial ϕ . Moreover*

$$(4.4) \quad \mathbf{u} \in C^\infty(\mathbb{R}^n \times [0, T_0]).$$

Proof. In (4.1), for short, we suppress (t, x) -variables and set

$$L_{\lambda, \mu} \mathbf{u} := -\mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u}, \quad \Phi(\mathbf{u}) = (\mathbf{u} \cdot \nabla) \mathbf{u}.$$

We convert (4.1) to the integral equation

$$(4.5) \quad \mathbf{u}(t) = e^{-tL_{\lambda, \mu}} \phi + \int_0^t e^{-(t-s)L_{\lambda, \mu}} \Phi(\mathbf{u}(s)) ds.$$

This equality is actually an equivalent expression of (3.13) for the solution if we replace $\mathbf{f}(\mathbf{y}, \tau)$ by $-\Phi(\mathbf{u}(s))$ (Duhamel's principle, see, p. 24 and p. 27 of [44], or p. 314 of [45]). Denote

$$(4.6) \quad \Psi \mathbf{u}(t) := e^{-tL_{\lambda, \mu}} \phi + \int_0^t e^{-(t-s)L_{\lambda, \mu}} \Phi(\mathbf{u}(s)) ds,$$

and put

$$X = \mathbf{H}^{m+1}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n), \quad Y = \mathbf{H}^m(\mathbb{R}^n) \cap C_0(\mathbb{R}^n), \quad m > n/2.$$

From (3.46) of Theorem 3.5, we have that

$$(4.7) \quad \|e^{-tL_{\lambda, \mu}} \mathbf{h}\|_{\mathbf{H}^{m+1}(\mathbb{R}^n)} \leq C_1 t^{-1/2} \|\mathbf{h}\|_{\mathbf{H}^m(\mathbb{R}^n)} \quad \text{for any } \mathbf{h} \in \mathbf{H}^m(\mathbb{R}^n), \quad 0 < t \leq 1,$$

where the constant C_1 is independent of λ , and depends only on the Lamé constant $\mu > 0$. Note that $\|\mathbf{h}\|_{\mathbf{H}^m(\mathbb{R}^n)} \leq \|\mathbf{h}\|_{\mathbf{H}^{m+1}(\mathbb{R}^n)}$ for any $\mathbf{h} \in \mathbf{H}^{m+1}(\mathbb{R}^n)$, and that $\mathbf{H}^{m+1}(\mathbb{R}^n)$, $(m > \frac{n}{2})$, can be compactly imbedded in $C_0^1(\mathbb{R}^n)$. This implies $(\mathbf{u} \cdot \nabla) \mathbf{u} - (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} = (\mathbf{u} \cdot \nabla)(\mathbf{u} - \tilde{\mathbf{u}}) + ((\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla) \tilde{\mathbf{u}} \in C([0, T]; \mathbf{H}^m(\mathbb{R}^n))$ for any $\mathbf{u}, \tilde{\mathbf{u}} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n))$. Then, the following four conditions hold:

- (a) $e^{-tL_{\lambda, \mu}} : X \rightarrow X$ is a strongly continuous semigroup, for $t \geq 0$;
- (b) $\Phi : X \rightarrow Y$ is Lipschitz, uniformly on bounded sets;
- (c) $e^{-tL_{\lambda, \mu}} : Y \rightarrow X$, for $t > 0$;
- (d) $\|e^{-tL_{\lambda, \mu}}\|_{\mathcal{L}(Y, X)} \leq \frac{C_1}{t^{1/2}}$, for $t \in (0, 1]$.

It is easy to see that Ψ acts on $C([0, T]; X)$, for each $T > 0$. Fix $\sigma > 0$, and set

$$(4.8) \quad Z = \{\mathbf{u} \in C([0, T]; X) \mid \mathbf{u}(0) = \phi, \|\mathbf{u}(t) - \phi\|_X \leq \sigma\}.$$

We want to pick T small enough that $\Phi : Z \rightarrow Z$ is a contraction mapping. By condition (a), we can choose T_1 so that $\|e^{-tL_{\lambda,\mu}}\phi - \phi\|_X \leq \sigma/2$ for $t \in [0, T_1]$. Now, if $\mathbf{u} \in Z$, then, by condition (b), we have a bound $\|\Phi(\mathbf{u}(s))\|_Y \leq K_1$, for $s \in [0, T_1]$, so, using condition (d), we obtain

$$(4.9) \quad \left\| \int_0^t e^{-(t-s)L_{\lambda,\mu}} \Phi(\mathbf{u}(s)) ds \right\|_X \leq C_1 t^{1/2} K_1.$$

If we pick $T_2 \leq T_1$ small enough, this will be $\leq \sigma/2$ for $t \in [0, T_2]$; therefore $\Psi : Z \rightarrow Z$, provided $T \leq T_2$. To arrange that Ψ be a contraction, we again use condition (b) to get

$$\|\Phi(\mathbf{u}(s)) - \Phi(\tilde{\mathbf{u}}(s))\|_Y \leq K \|\mathbf{u}(s) - \tilde{\mathbf{u}}(s)\|_X,$$

for $\mathbf{u}, \tilde{\mathbf{u}} \in Z$. Thus, for $t \in [0, T_2]$,

$$(4.10) \quad \begin{aligned} \|\Psi(\mathbf{u})(t) - \Psi(\tilde{\mathbf{u}})(t)\|_X &= \left\| \int_0^t e^{-(t-s)L_{\lambda,\mu}} [\Phi(\mathbf{u}(s)) - \Phi(\tilde{\mathbf{u}}(s))] ds \right\|_X \\ &\leq C_1 t^{1/2} K \sup \|\mathbf{u}(s) - \tilde{\mathbf{u}}(s)\|_X; \end{aligned}$$

and now if $T := T_0 \leq T_2$ is chosen small enough, we obtain $C_1 T_0^{1/2} K < 1$, making Ψ a contraction mapping on Z . Hence Ψ has a unique fixed point $\mathbf{u}$ in Z , solving (4.5). In other words, $\mathbf{u} \in Z$ satisfies

$$(4.11) \quad \mathbf{u}(t) = e^{-tL_{\lambda,\mu}}\phi + \int_0^t e^{-(t-s)L_{\lambda,\mu}} \Phi(\mathbf{u}(s)) ds, \quad 0 \leq t \leq T_0,$$

or equivalently,

$$(4.12) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, \quad 0 \leq t \leq T_0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Clearly, $\mathbf{u} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$. Recalling that there is a continuously (compact) imbedding $\mathbf{H}^{m+1}(\mathbb{R}^n) \rightarrow C_0^1(\mathbb{R}^n)$ for $m > n/2$, we thus have $\mathbf{u}, \nabla \mathbf{u} \in C([0, T_0]; C_0(\mathbb{R}^n))$. Let us write (4.12) as

$$(4.13) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{F}(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^n, \quad 0 \leq t \leq T_0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n \end{cases}$$

with $\mathbf{F}(\mathbf{x}, t) := -(\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t)$. Applying (4.3) of Lemma 4.1, we know $(\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \in C([0, T_0]; \mathbf{H}^m(\mathbb{R}^n))$. By this and observing the equation (4.12), we immediately see that $\frac{\partial \mathbf{u}}{\partial t} \in C([0, T_0]; \mathbf{H}^{m-1}(\mathbb{R}^n))$. It follows from the regularity of linear parabolic system (cf. p. 384-388 of [11]) that $\mathbf{u} \in L^2([0, T_0]; \mathbf{H}^{m+2}(\mathbb{R}^n)) \cap C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$ and $\frac{\partial \mathbf{u}}{\partial t} \in L^2([0, T_0]; \mathbf{H}^m(\mathbb{R}^n))$. In view of $\phi \in \mathcal{S}(\mathbb{R}^n)$, we get that $\partial_{\mathbf{x}}^\alpha \mathbf{u}(\mathbf{x}, 0) \in C_0(\mathbb{R}^n)$ for all $|\alpha| = m > n/2$, so, we find by the compatibility condition (cf. p. 388 of [11]) that the smoothness of the t -derivatives can be extended up to $t = 0$. We can iterate (or bootstrap) this argument repeatedly, and also, via the system of equations (4.13), to obtain the regularity of $\mathbf{x}$ -derivatives and t -derivatives of $\mathbf{u}$, i.e., $\mathbf{u} \in C^\infty(\mathbb{R}^n \times [0, T_0])$ (see, p. 382-390 of [11]).

Suppose there exists another solution $\tilde{\mathbf{u}} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$ satisfying (4.12). Then we have

$$(4.14) \quad \begin{cases} \frac{\partial(\mathbf{u}-\tilde{\mathbf{u}})}{\partial t} - \mu \Delta(\mathbf{u}-\tilde{\mathbf{u}}) - (\lambda + \mu) \operatorname{grad} \operatorname{div}(\mathbf{u}-\tilde{\mathbf{u}}) + (\mathbf{u} \cdot \nabla)(\mathbf{u}-\tilde{\mathbf{u}}) + (\mathbf{u}-\tilde{\mathbf{u}}) \cdot \nabla \tilde{\mathbf{u}} = 0, & \mathbf{x} \in \mathbb{R}^n, \quad 0 \leq t \leq T_0, \\ (\mathbf{u}-\tilde{\mathbf{u}})|_{t=0} = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

This is an initial-value problem of linear parabolic system for unknown vector-valued function $\mathbf{u} - \tilde{\mathbf{u}}$ with vanishing initial-value. Since $\mathbf{u}$ and $\nabla \tilde{\mathbf{u}}$ is continuous and bounded in $[0, T_0] \times \mathbb{R}^n$, and Hölder continuous (exponent α) in $\mathbf{x}$, uniformly with respect to $(\mathbf{x}, t)$ in bounded subsets of $\mathbb{R}^n \times [0, T_0]$, we immediately see that (4.14) has a unique solution $\mathbf{u} - \tilde{\mathbf{u}} = 0$ by (3.13) and Friedman's inequality (3.15) (with $\alpha = 0$). Thus, we have $\mathbf{u} = \tilde{\mathbf{u}} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$. $\square$

In the proof of Theorem 4.2, we have shown that the solution $\mathbf{u}$ of (4.12) is continuous bounded on $\mathbb{R}^n \times [0, T_0]$, where T_0 is estimable from below in terms of $\|\phi\|_{\mathbf{H}^m(\mathbb{R}^n)}$. But this bound is only valid in a short time interval $[0, T_0]$. Our purpose is to find larger time interval $[0, T]$ in which the solution still exists with $T > T_0$. The following theorem (the so-called “maximum principle” for the solutions of the parabolic inertial Lamé equations in three-dimensional Euclidean space) is a stronger uniform control estimate, which will play a key role in our arguments:

Theorem 4.3. *Let $\mu > 0$ and $\lambda + \mu \geq 0$, and suppose $\phi \in \mathcal{S}(\mathbb{R}^3)$. If $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^3)) \cap \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{C}^\infty(\mathbb{R}^3 \times [0, T])$, $m > \frac{3}{2}$, is the unique local smooth solution of*

$$(4.15) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

with $0 < T \leq \infty$, then

$$(4.16) \quad \|\mathbf{u}(\mathbf{x}, t)\|_{\mathbf{C}(\mathbb{R}^3 \times [0, T])} \leq C \|\phi\|_{\mathbf{C}(\mathbb{R}^3)} \text{ for all } 0 \leq t \leq T,$$

where the constant C is independent of λ and T , and depends only on $\mu > 0$.

Proof. 1. Set $\mathcal{V} := \{\mathbf{g} \in \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{C}^\infty(\mathbb{R}^3) \mid \operatorname{div} \mathbf{g} = 0\}$. Clearly, $\mathcal{V}$ is a linear space. Then, set

$$\mathbf{V}^j = \text{closure of } \mathcal{V} \text{ in } \mathbf{H}^j(\mathbb{R}^3), \quad j = 0, 1, 2, \dots, m+1.$$

It follows from the Hodge decomposition (see, p.553 of [45] or p.108 of [25]) that $\mathbf{L}^2(\mathbb{R}^3)$ can be decomposed into the direct sum of the following two spaces:

$$\mathbf{L}^2(\mathbb{R}^3) = \mathbf{V}^0 \oplus (\mathbf{V}^0)^\perp,$$

where

$$(\mathbf{V}^0)^\perp = \{\mathbf{k} \mid \mathbf{k} = \operatorname{grad} p, p \in H^1(\mathbb{R}^3)\}.$$

Obviously, $\mathbf{V}^0$ and $(\mathbf{V}^0)^\perp$, the two subspaces of $\mathbf{L}^2(\mathbb{R}^3)$, are orthogonal on each other (see, for example, Proposition 1.16 of [34]). From the well-known identities

$$(4.17) \quad \operatorname{div} \operatorname{curl} = 0, \quad \operatorname{curl} \operatorname{grad} = 0,$$

we also see (cf. Lemma 1.1 of Chapter 1 of [25]) that $\mathbf{V}^0 = \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\mathbb{R}^3)$, where $\mathcal{V} := \{\mathbf{h} \mid \mathbf{h} = \operatorname{curl} \mathbf{q}, \mathbf{q} \in \mathbf{C}_0^1(\mathbb{R}^3) \cap \mathbf{C}^\infty(\mathbb{R}^3)\}$.

Since $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^3) \cap \mathbf{C}_0(\mathbb{R}^3)) \cap \mathbf{C}^\infty(\mathbb{R}^3 \times [0, T])$, $m > \frac{3}{2}$ is the unique local smooth solution of (4.15), it follows that $\mathbf{u}$ is just the unique smooth solution of the following linear parabolic system with unknown vector field $\tilde{\mathbf{u}}$:

$$(4.18) \quad \begin{cases} \frac{\partial \tilde{\mathbf{u}}}{\partial t} - \mu \Delta \tilde{\mathbf{u}} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \tilde{\mathbf{u}} + (\mathbf{u} \cdot \nabla) \tilde{\mathbf{u}} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \tilde{\mathbf{u}}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Applying the Hodge decomposition again (see, for example, Lemma 1.3 and Lemma 1.4 (on p.59) of Chapter 1 of [25]) that the solution $\tilde{\mathbf{u}} = \mathbf{u}$ of (4.18) can be uniquely decomposed into the superposition of a transverse field and a longitudinal field:

$$(4.19) \quad \tilde{\mathbf{u}} = \mathbf{u} = \mathbf{v} + \mathbf{w},$$

where $\mathbf{v} \in C^\infty([0, T]; \mathbf{C}_0^\infty(\mathbb{R}^3))$ and $\mathbf{w} \in C^\infty([0, T]; \mathbf{C}_0^\infty(\mathbb{R}^3))$ are the transverse and longitudinal fields, respectively, i.e., they satisfy the conditions

$$(4.20) \quad \operatorname{div} \mathbf{v} = \mathbf{0}, \quad \operatorname{curl} \mathbf{w} = \mathbf{0}.$$

This decomposition corresponds to uniquely decomposing the elastic vector field into an isometric expansion part and an irrotational part. In particular,

$$(4.21) \quad \phi = \phi_1 + \phi_2 = \mathbf{v}(0) + \mathbf{w}(0), \quad \operatorname{div} \phi_1 = 0 \quad \text{and} \quad \operatorname{curl} \phi_2 = 0,$$

where $\phi_1, \phi_2 \in C_0(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$. Substituting (4.19) into (4.18), we find by (4.20) that

$$(4.22) \quad \begin{cases} \frac{\partial(\mathbf{v}+\mathbf{w})}{\partial t} - \mu\Delta(\mathbf{v}+\mathbf{w}) - (\lambda+\mu)\operatorname{grad} \operatorname{div} \mathbf{w} + (\mathbf{u} \cdot \nabla)(\mathbf{v}+\mathbf{w}) = 0, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \mathbf{u}(\mathbf{x}, 0) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

By Lemma 1.2 of Chapter 1 of [25], the longitudinal field $\mathbf{w}$ can be expressed by the gradient of a scalar field

$$(4.23) \quad \mathbf{w} = \operatorname{grad} p,$$

so that (4.22) can be rewritten as

$$(4.24) \quad \begin{cases} \left[\frac{\partial \mathbf{v}}{\partial t} - \mu\Delta \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} \right] + \left[\frac{\partial \mathbf{w}}{\partial t} - (\lambda + 2\mu)\Delta \mathbf{w} + (\mathbf{u} \cdot \nabla) \mathbf{w} \right] = 0, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \operatorname{div} \mathbf{v} = 0, \ \operatorname{curl} \mathbf{w} = 0, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \mathbf{v}(\mathbf{x}, 0) + \mathbf{w}(\mathbf{x}, 0) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Clearly, (4.24) can be equivalently expressed as two systems:

$$(4.25) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \mu\Delta \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} = \mathbf{f}, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \operatorname{div} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \mathbf{v}(\mathbf{x}, 0) = \phi_1(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 \end{cases}$$

and

$$(4.26) \quad \begin{cases} \frac{\partial \mathbf{w}}{\partial t} - (\lambda + 2\mu)\Delta \mathbf{w} + (\mathbf{u} \cdot \nabla) \mathbf{w} = -\mathbf{f}, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \operatorname{curl} \mathbf{w} = 0, & \mathbf{x} \in \mathbb{R}^3, \ 0 \leq t \leq T, \\ \mathbf{w}(\mathbf{x}, 0) = \phi_2(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

where $\mathbf{f}$ is some vector field on $\mathbb{R}^3 \times [0, T]$.

2. We will further determine the vector field $\mathbf{f}$. Denote by P the projection operator from $\mathbf{L}^2(\mathbb{R}^3)$ to $\mathbf{V}^0$:

$$Pr = \mathbf{h}, \quad \forall r \in \mathbf{L}^2(\mathbb{R}^3),$$

where $\mathbf{h} \in \mathbf{V}^0$ is given by the Hodge decomposition formula

$$\mathbf{r} = \mathbf{h} + \operatorname{grad} p,$$

with $\operatorname{div} \mathbf{h} = 0$ in $\mathbb{R}^3$. For the given unique solution $\mathbf{u}$ of (4.15), we denote $\Lambda_{\mathbf{u}}(\mathbf{h}) = (\mathbf{u} \cdot \nabla) \mathbf{h}$, $\forall \mathbf{h} \in \mathbf{V}^1 \cap C_0(\mathbb{R}^3)$. Let $\phi_1 \in C_0(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \cap \mathbf{V}^0$ be the transverse component of ϕ given in step 1. We consider the following integral equation:

$$(4.27) \quad \mathbf{h}(t) = e^{t(\mu\Delta)} \phi_1 + \int_0^t e^{(t-s)(\mu\Delta)} P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) ds.$$

We denote

$$\Xi_{\mathbf{u}} \mathbf{h}(t) := e^{t(\mu\Delta)} \phi_1 + \int_0^t e^{(t-s)(\mu\Delta)} P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) ds,$$

and put

$$X = \mathbf{V}^{m+1} \cap C_0(\mathbb{R}^3), \quad Y = \mathbf{V}^m \cap C_0(\mathbb{R}^3), \quad m > 3/2.$$

Recall that $\mathbf{u} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \subset C([0, T_0]; C_0(\mathbb{R}^n))$ for $m > n/2$. It follows that $P((\mathbf{u} \cdot \nabla) \mathbf{h}) - P((\mathbf{u} \cdot \nabla) \tilde{\mathbf{h}}) = P((\mathbf{u} \cdot \nabla)(\mathbf{h} - \tilde{\mathbf{h}})) \in C([0, T]; \mathbf{V}_0 \cap \mathbf{H}^m(\mathbb{R}^3))$ for any $\mathbf{h}, \tilde{\mathbf{h}} \in C([0, T]; \mathbf{V}_0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3))$, and the following four conditions hold:

- (a) $e^{t\mu\Delta} : X \rightarrow X$ is a strongly continuous semigroup, for $t \geq 0$;
- (b) $P \circ \Lambda_{\mathbf{u}} : X \rightarrow Y$ is Lipschitz, uniformly on bounded sets;
- (c) $e^{t\mu\Delta} : Y \rightarrow X$, for $t > 0$;
- (d) $\|e^{t\mu\Delta}\|_{\mathcal{L}(Y, X)} \leq \frac{C_1}{t^{1/2}}$, for $t \in (0, 1]$.

Obviously, $\Xi_{\mathbf{u}}$ acts on $C([0, T]; X)$, for each $T > 0$. Fix $\sigma > 0$, and set

$$(4.28) \quad Z = \{\mathbf{h} \in C([0, T]; X) \mid \mathbf{h}(0) = \phi_1, \|\mathbf{h}(t) - \phi_1\|_X \leq \sigma\}.$$

By condition (a), we can choose T_1 so that $\|e^{t\mu\Delta}\phi_1 - \phi_1\|_X \leq \sigma/2$ for $t \in [0, T_1]$. If $\mathbf{h} \in Z$, then, by condition (b), we have a bound $\|P(\Lambda_{\mathbf{u}}(\mathbf{h}(s)))\|_Y \leq K_1$, for $s \in [0, T_1]$, so, using condition (d), we get

$$(4.29) \quad \left\| \int_0^t e^{(t-s)(\mu\Delta)} P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) ds \right\|_X \leq C_1 t^{1/2} K_1.$$

If we pick $T_2 \leq T_1$ small enough, this will be $\leq \sigma/2$ for $t \in [0, T_2]$; therefore $\Xi_{\mathbf{u}} : Z \rightarrow Z$, provided $T \leq T_2$. We again use condition (b) to get

$$\|P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) - P(\Lambda_{\mathbf{u}}(\tilde{\mathbf{h}}(s)))\|_Y \leq K \|\mathbf{h}(s) - \tilde{\mathbf{h}}(s)\|_X,$$

for $\mathbf{h}, \tilde{\mathbf{h}} \in Z$. Thus, for $t \in [0, T_2]$,

$$(4.30) \quad \begin{aligned} \|\Xi_{\mathbf{u}}(\mathbf{h})(t) - \Xi_{\mathbf{u}}(\tilde{\mathbf{h}})(t)\|_X &= \left\| \int_0^t e^{(t-s)(\mu\Delta)} [P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) - P(\Lambda_{\mathbf{u}}(\tilde{\mathbf{h}}(s)))] ds \right\|_X \\ &\leq C_1 t^{1/2} K \sup \|\mathbf{h}(s) - \tilde{\mathbf{h}}(s)\|_X; \end{aligned}$$

and now if $T := T_0 \leq T_2$ is chosen small enough, we obtain $C_1 T_0^{1/2} K < 1$, making $\Xi_{\mathbf{u}}$ a contraction mapping on Z . Hence $\Xi_{\mathbf{u}}$ has a unique fixed point $\mathbf{h}$ in Z , solving (4.27). In other words, $\mathbf{h} \in Z$ satisfies

$$(4.31) \quad \mathbf{h}(t) = e^{t(\mu\Delta)}\phi_1 + \int_0^t e^{(t-s)(\mu\Delta)} P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) ds, \quad 0 \leq t \leq T_0.$$

Since $\operatorname{div}(e^{t(\mu\Delta)}\phi_1) = 0$ and $\operatorname{div}(\int_0^t e^{(t-s)(\mu\Delta)} P(\Lambda_{\mathbf{u}}(\mathbf{h}(s))) ds) = 0$ in $\mathbb{R}^3$ for each $t \geq 0$, and $\mathbf{h} \in C^1([0, T_0]; \mathbf{V}^0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3))$, it follows that

$$P \frac{\partial \mathbf{h}}{\partial t} = \frac{\partial \mathbf{h}}{\partial t},$$

and thus (4.31) is equivalent to

$$(4.32) \quad \frac{\partial \mathbf{h}}{\partial t} - P[\mu\Delta\mathbf{h} - (\mathbf{u} \cdot \nabla)\mathbf{h}] = 0, \quad \mathbf{h} \in C([0, T_0]; \mathbf{V}^0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3)).$$

According to the Hodge decomposition, there exists a $p \in C([0, T_0]; H^1(\mathbb{R}^3))$ such that

$$(4.33) \quad P[\mu\Delta\mathbf{h} - (\mathbf{u} \cdot \nabla)\mathbf{h}] + \nabla p = \mu\Delta\mathbf{h} - (\mathbf{u} \cdot \nabla)\mathbf{h}.$$

Therefore, we get

$$(4.34) \quad \begin{cases} \frac{\partial \mathbf{h}}{\partial t} - \mu\Delta\mathbf{h} + (\mathbf{u} \cdot \nabla)\mathbf{h} = -\nabla p, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{div} \mathbf{h} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \mathbf{h}(0, \mathbf{x}) = \phi_1(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Applying the regularity of linear parabolic system, we get $\mathbf{h} \in C^1([0, T_0]; \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{V}^0) \cap \mathbf{C}^\infty(\mathbb{R}^3 \times [0, T_0])$. In view of the fixed point theorem of the Banach contraction mapping, it follows that $(\mathbf{h}, \nabla p)$ is the unique smooth solution of (4.34) with $\mathbf{h} \in \mathbf{V}^0$, $\nabla p \in (\mathbf{V}^0)^\perp$. Subtracting (4.34) from (4.25) gives

$$(4.35) \quad \begin{cases} \frac{\partial \boldsymbol{\theta}}{\partial t} - \mu\Delta\boldsymbol{\theta} + (\mathbf{u} \cdot \nabla)\boldsymbol{\theta} = \mathbf{f} + \nabla p, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{div} \boldsymbol{\theta} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \boldsymbol{\theta}(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

where $\boldsymbol{\theta} = \mathbf{v} - \mathbf{h}$. It is clear that there exists a unique $\nabla \tilde{p} \in (\mathbf{V}^0)^\perp$ such that

$$P \left[\frac{\partial \boldsymbol{\theta}}{\partial t} - \mu\Delta\boldsymbol{\theta} + (\mathbf{u} \cdot \nabla)\boldsymbol{\theta} \right] + \nabla \tilde{p} = \frac{\partial \boldsymbol{\theta}}{\partial t} - \mu\Delta\boldsymbol{\theta} + (\mathbf{u} \cdot \nabla)\boldsymbol{\theta},$$

and hence (4.35) can be equivalently written as

$$(4.36) \quad \begin{cases} P\left[\frac{\partial \boldsymbol{\theta}}{\partial t} - \mu \Delta \boldsymbol{\theta} + (\mathbf{u} \cdot \nabla) \boldsymbol{\theta}\right] = \mathbf{f} + \nabla p - \nabla \tilde{p}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{div} \boldsymbol{\theta} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \boldsymbol{\theta}(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

Note that the adjoint equation of

$$\mathcal{L}\boldsymbol{\theta} := \frac{\partial \boldsymbol{\theta}}{\partial t} - \mu \Delta \boldsymbol{\theta} + (\mathbf{u} \cdot \nabla) \boldsymbol{\theta} = 0$$

is

$$\mathcal{L}^* \boldsymbol{\eta} := \frac{\partial \boldsymbol{\eta}}{\partial t} + \mu \Delta \boldsymbol{\eta} - (\mathbf{u} \cdot \nabla) \boldsymbol{\eta} - (\operatorname{div} \mathbf{u}) \boldsymbol{\eta} = 0,$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. Furthermore, (4.36) can be rewritten as

$$(4.37) \quad \begin{cases} P\mathcal{L}(\boldsymbol{\theta}) = \mathbf{f} + \nabla p - \nabla \tilde{p}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{div} \boldsymbol{\theta} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \boldsymbol{\theta}(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

In view of $(P\mathcal{L})^* = \mathcal{L}^*P^* = \mathcal{L}^*P$, we see that the null-space of the operator $\mathcal{L}^*P$ is just $\mathbf{V}^0$. Note also that the null-space of the operator $\mathcal{L}^*P$ is exactly the all solutions of the following homogeneous linear equations

$$(4.38) \quad \begin{cases} \mathcal{L}^*P(\boldsymbol{\eta}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{div} \boldsymbol{\eta} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \boldsymbol{\eta}(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Since

$$(4.39) \quad \begin{aligned} \langle P\mathcal{L}(\boldsymbol{\theta}), \boldsymbol{\eta} \rangle_{L^2(\mathbb{R}^3)} &= \langle \boldsymbol{\theta}, (P\mathcal{L})^* \boldsymbol{\eta} \rangle_{L^2(\mathbb{R}^3)} \\ &= \langle \boldsymbol{\theta}, \mathcal{L}^*P(\boldsymbol{\eta}) \rangle_{L^2(\mathbb{R}^3)} \quad \text{for all } \boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbf{H}^{m+1}(\mathbb{R}^3) \cap \mathbf{C}_0(\mathbb{R}^3), \end{aligned}$$

we find from this and $\mathcal{L}^*P(\boldsymbol{\eta}) = \mathbf{0}$ (see (4.38)) that

$$(4.40) \quad \langle P\mathcal{L}(\boldsymbol{\theta}), \boldsymbol{\eta} \rangle_{L^2(\mathbb{R}^3)} = 0 \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}^0,$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n)}$ denotes the inner product in $L^2(\mathbb{R}^n)$. This and (4.37) imply that the vector field $\mathbf{f} + \nabla p - \nabla \tilde{p}$ is orthogonal to $\mathbf{V}^0$ for each $t \in [0, T_0]$, i.e., $\mathbf{f} \in C([0, T_0]; (\mathbf{V}^0)^\perp)$ because $\nabla p - \nabla \tilde{p} \in (\mathbf{V}^0)^\perp$. In addition, since (4.34) is a system of linear parabolic equations, the solution $\mathbf{h}$ of (4.34) can be uniquely extended from $\mathbb{R}^3 \times [0, T_0]$ to $\mathbb{R}^3 \times [0, T]$ by the standard linear parabolic theory, where the known vector field $\mathbf{u}$ is defined. Therefore,

$$(4.41) \quad \mathbf{f} \in C([0, T]; (\mathbf{V}^0)^\perp).$$

3. Define $\mathcal{W} := \{\mathbf{l} \in \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{C}^\infty(\mathbb{R}^3) \mid \operatorname{curl} \mathbf{l} = \mathbf{0}\}$. Clearly, $\mathcal{W}$ is a linear space. Then, set

$$\mathbf{W}^j = \text{closure of } \mathcal{W} \text{ in } \mathbf{H}^j(\mathbb{R}^3), \quad j = 0, 1, 2, \dots, m+1.$$

From (4.17), it follows that $\mathbf{W}^0 = (\mathbf{V}^0)^\perp$. For the given solution $\mathbf{u}$ of (4.15), define $\Pi_{\mathbf{u}}(\mathbf{k}) = (\mathbf{u} \cdot \nabla) \mathbf{k}$ for any $\mathbf{k} \in C([0, T]; \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{H}^{m+1}(\mathbb{R}^3) \cap \mathbf{W}^0)$. Let $\boldsymbol{\phi}_2 \in \mathbf{C}_0(\mathbb{R}^3) \cap \mathbf{C}^\infty(\mathbb{R}^3) \cap \mathbf{W}^0$ be the longitudinal component of $\boldsymbol{\phi}$ given in step 1. Consider the integral equation:

$$(4.42) \quad \mathbf{k}(t) = e^{t(\lambda+2\mu)\Delta} \boldsymbol{\phi}_2 + \int_0^t e^{(t-s)(\lambda+2\mu)\Delta} Q(\Pi_{\mathbf{u}}(\mathbf{k}(s))) ds,$$

where Q is the projection operator from $L^2(\mathbb{R}^3)$ to $\mathbf{W}^0$:

$$Q\mathbf{r} = \mathbf{k}, \quad \forall \mathbf{r} \in L^2(\mathbb{R}^3),$$

where $\mathbf{k} \in \mathbf{W}^0$ given by the Hodge decomposition:

$$\mathbf{r} = \operatorname{curl} q + \mathbf{k}$$

and $\operatorname{curl} \mathbf{k} = 0$ in $\mathbb{R}^3$. Define the operator

$$\Theta_{\mathbf{u}} \mathbf{k}(t) := e^{t(\lambda+2\mu)\Delta} \phi_2 + \int_0^t e^{(t-s)(\lambda+2\mu)\Delta} Q[\Pi_{\mathbf{u}}(\mathbf{k}(s))] ds,$$

and introduce the spaces

$$X = \mathbf{W}^{m+1} \cap C_0(\mathbb{R}^3), \quad Y = \mathbf{W}^m \cap C_0(\mathbb{R}^3), \quad m > 3/2.$$

It follows that $Q[(\mathbf{u} \cdot \nabla) \mathbf{k}] - Q[(\mathbf{u} \cdot \nabla) \tilde{\mathbf{k}}] = Q[(\mathbf{u} \cdot \nabla)(\mathbf{k} - \tilde{\mathbf{k}})] \in C([0, T]; \mathbf{W}_0 \cap \mathbf{H}^m(\mathbb{R}^3))$ for any $\mathbf{k}, \tilde{\mathbf{k}} \in C([0, T]; \mathbf{W}_0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3))$, and the following four conditions hold:

- (a') $e^{t(\lambda+2\mu)\Delta} : X \rightarrow X$ is a strongly continuous semigroup for $t \geq 0$;
- (b') $Q \circ \Pi_{\mathbf{u}} : X \rightarrow Y$ is Lipschitz, uniformly on bounded sets;
- (c') $e^{t(\lambda+2\mu)\Delta} : Y \rightarrow X$, for $t > 0$;
- (d') $\|e^{t(\lambda+2\mu)\Delta}\|_{\mathcal{L}(Y, X)} \leq \frac{C_1}{t^{1/2}}$, for $t \in (0, 1]$.

Clearly, $\Theta_{\mathbf{u}}$ acts on $C([0, T]; X)$, for each $T > 0$. Fix $\sigma > 0$, and define

$$(4.43) \quad Z = \{\mathbf{k} \in C([0, T]; X) \mid \mathbf{k}(0) = \phi_2, \|\mathbf{k}(t) - \phi_2\|_X \leq \sigma\}.$$

By condition (a'), we can choose T_1 so that $\|e^{t(\lambda+2\mu)\Delta} \phi_2 - \phi_2\|_X \leq \sigma/2$ for all $t \in [0, T_1]$. If $\mathbf{k} \in Z$, then, by condition (b'), we find a bound $\|Q[\Pi_{\mathbf{u}}(\mathbf{k}(s))]\|_Y \leq K_1$ for $s \in [0, T_1]$. Using condition (d'), we obtain

$$(4.44) \quad \left\| \int_0^t e^{(t-s)(\lambda+2\mu)\Delta} Q[\Pi_{\mathbf{u}}(\mathbf{k}(s))] ds \right\|_X \leq C_1 t^{1/2} K_1.$$

If we choose $T_2 \leq T_1$ sufficiently small, this bound becomes $\leq \sigma/2$ for all $t \in [0, T_2]$; therefore $\Theta_{\mathbf{u}} : Z \rightarrow Z$, provided $T \leq T_2$. Applying condition (b') again yields

$$\|Q(\Pi_{\mathbf{u}}(\mathbf{k}(s))) - Q(\Pi_{\mathbf{u}}(\tilde{\mathbf{k}}(s)))\|_Y \leq K \|\mathbf{k}(s) - \tilde{\mathbf{k}}(s)\|_X,$$

for any $\mathbf{k}, \tilde{\mathbf{k}} \in Z$. Thus, for $t \in [0, T_2]$,

$$(4.45) \quad \begin{aligned} \|\Theta_{\mathbf{u}}(\mathbf{k})(t) - \Theta_{\mathbf{u}}(\tilde{\mathbf{k}})(t)\|_X &= \left\| \int_0^t e^{(t-s)(\lambda+2\mu)\Delta} [Q(\Pi_{\mathbf{u}}(\mathbf{k}(s))) - Q(\Pi_{\mathbf{u}}(\tilde{\mathbf{k}}(s)))] ds \right\|_X \\ &\leq C_1 t^{1/2} K \sup \|\mathbf{k}(s) - \tilde{\mathbf{k}}(s)\|_X. \end{aligned}$$

Now, if we take $T := T_0 \leq T_2$ sufficiently small so that $C_1 T_0^{1/2} K < 1$, then $\Theta_{\mathbf{u}}$ becomes a contraction mapping on Z . Hence $\Theta_{\mathbf{u}}$ admits a unique fixed point $\mathbf{k} \in Z$, which solves (4.42). That is, $\mathbf{k} \in Z$ satisfies

$$(4.46) \quad \mathbf{k}(t) = e^{t(\lambda+2\mu)\Delta} \phi_2 + \int_0^t e^{(t-s)(\lambda+2\mu)\Delta} Q(\Pi_{\mathbf{u}}(\mathbf{k}(s))) ds, \quad 0 \leq t \leq T_0.$$

Since $\operatorname{curl}(e^{t(\lambda+2\mu)\Delta} \phi_2) = 0$ and $\operatorname{curl}(\int_0^t e^{(t-s)(\lambda+2\mu)\Delta} Q(\Pi_{\mathbf{u}}(\mathbf{k}(s))) ds) = 0$ in $\mathbb{R}^3$ for each $t \geq 0$, and $\mathbf{k} \in C([0, T_0]; \mathbf{W}^0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3))$, it follows that

$$Q \frac{\partial \mathbf{k}}{\partial t} = \frac{\partial \mathbf{k}}{\partial t},$$

and thus (4.46) is equivalent to

$$(4.47) \quad \frac{\partial \mathbf{k}}{\partial t} - Q[(\lambda + 2\mu)\Delta \mathbf{k} - (\mathbf{u} \cdot \nabla) \mathbf{k}] = 0, \quad \mathbf{k} \in C([0, T_0]; \mathbf{W}^0 \cap \mathbf{H}^{m+1}(\mathbb{R}^3)).$$

According to the Hodge decomposition, there exists a $\mathbf{q} \in C([0, T_0]; \mathbf{H}^1(\mathbb{R}^3))$ such that

$$(4.48) \quad Q[(\lambda + 2\mu)\Delta \mathbf{k} - (\mathbf{u} \cdot \nabla)\mathbf{k}] + \operatorname{curl} \mathbf{q} = (\lambda + 2\mu)\Delta \mathbf{k} - (\mathbf{u} \cdot \nabla)\mathbf{k}.$$

Therefore, we obtain the system:

$$(4.49) \quad \begin{cases} \frac{\partial \mathbf{k}}{\partial t} - (\lambda + 2\mu)\Delta \mathbf{k} + (\mathbf{u} \cdot \nabla)\mathbf{k} = -\operatorname{curl} \mathbf{q}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{curl} \mathbf{k} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \mathbf{k}(0, \mathbf{x}) = \phi_2(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Applying the regularity of linear parabolic system, we conclude that $\mathbf{k} \in C^1([0, T_0]; \mathbf{W}^0) \cap C^\infty(\mathbb{R}^3 \times [0, T_0])$. From the fixed point theorem of contraction mapping $\Theta_{\mathbf{u}}$, we deduce that the solution $\mathbf{k}$ of (4.34) is unique. By subtracting (4.49) from (4.26), we discover

$$(4.50) \quad \begin{cases} \frac{\partial \gamma}{\partial t} - (\lambda + 2\mu)\Delta \gamma + (\mathbf{u} \cdot \nabla)\gamma = -\mathbf{f} + \operatorname{curl} \mathbf{q}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{curl} \gamma = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \gamma(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

where $\gamma = \mathbf{w} - \mathbf{k}$. There exists a unique $\operatorname{curl} \tilde{\mathbf{q}} \in (\mathbf{W}^0)^\perp$ such that

$$Q\left[\frac{\partial \gamma}{\partial t} - (\lambda + 2\mu)\Delta \gamma + (\mathbf{u} \cdot \nabla)\gamma\right] + \operatorname{curl} \tilde{\mathbf{q}} = \frac{\partial \gamma}{\partial t} - (\lambda + 2\mu)\Delta \gamma + (\mathbf{u} \cdot \nabla)\gamma,$$

and hence (4.50) can be equivalently written as

$$(4.51) \quad \begin{cases} Q\left[\frac{\partial \gamma}{\partial t} - (\lambda + 2\mu)\Delta \gamma + (\mathbf{u} \cdot \nabla)\gamma\right] = -\mathbf{f} + \operatorname{curl} \mathbf{q} - \operatorname{curl} \tilde{\mathbf{q}}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{curl} \gamma = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \gamma(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

Recall that the adjoint equation of

$$\mathcal{L}\gamma := \frac{\partial \gamma}{\partial t} - (\lambda + 2\mu)\Delta \gamma + (\mathbf{u} \cdot \nabla)\gamma = 0$$

is

$$\mathcal{L}^*\beta := \frac{\partial \beta}{\partial t} + (\lambda + 2\mu)\Delta \beta - (\mathbf{u} \cdot \nabla)\beta - (\operatorname{div} \mathbf{u})\beta = 0,$$

where $\mathcal{L}^*$ denotes the adjoint operator of $\mathcal{L}$. Furthermore, (4.51) can be rewritten as

$$(4.52) \quad \begin{cases} Q\mathcal{L}(\gamma) = -\mathbf{f} + \operatorname{curl} \mathbf{q} - \operatorname{curl} \tilde{\mathbf{q}}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{curl} \gamma = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \gamma(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

Due to the fact that $(Q\mathcal{L})^* = \mathcal{L}^*Q^* = \mathcal{L}^*Q$, the null space of the operator $\mathcal{L}^*Q$ is precisely $\mathbf{W}^0$. Note that the null space of $\mathcal{L}^*Q$ consists of all solutions to the following homogeneous linear system:

$$(4.53) \quad \begin{cases} \mathcal{L}^*Q(\beta) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \operatorname{curl} \beta = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T_0, \\ \beta(0, \mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Since

$$(4.54) \quad \langle Q\mathcal{L}(\gamma), \beta \rangle_{L^2(\mathbb{R}^3)} = \langle \gamma, \mathcal{L}^*Q(\beta) \rangle_{L^2(\mathbb{R}^3)} \quad \text{for all } \gamma, \beta \in \mathbf{H}^{m+1}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3),$$

and $\mathcal{L}^*Q(\eta) = \mathbf{0}$ (see (4.53)), we conclude that

$$(4.55) \quad \langle Q\mathcal{L}(\gamma), \beta \rangle_{L^2(\mathbb{R}^3)} = 0 \quad \text{for all } \beta \in \mathbf{W}^0.$$

This implies that the vector field $\mathbf{f} + \operatorname{curl} \mathbf{q} - \operatorname{curl} \tilde{\mathbf{q}}$ is orthogonal to $\mathbf{W}^0$ for each $t \in [0, T_0]$, i.e., $\mathbf{f} \in C([0, T_0]; (\mathbf{W}^0)^\perp) = C([0, T_0]; \mathbf{V}^0)$. Moreover, since (4.49) is a system of linear parabolic

equations, its solution $\mathbf{k}$ can be uniquely extended from $\mathbb{R}^3 \times [0, T_0]$ to $\mathbb{R}^3 \times [0, T]$ by the standard linear parabolic theory, where the known vector field $\mathbf{u}$ is defined. Therefore,

$$(4.56) \quad -\mathbf{f} \in C([0, T]; (\mathbf{W}^0)^\perp) = C([0, T]; \mathbf{V}^0).$$

By virtue of $\mathbf{V}^0 \cap (\mathbf{V}^0)^\perp = \{\mathbf{0}\}$, we find by (4.41) and (4.56) that $\mathbf{f} \equiv \mathbf{0}$ in $\mathbb{R}^3 \times [0, T]$. In other words, if $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^3) \cap \mathbf{C}_0(\mathbb{R}^3)) \cap \mathbf{C}^\infty(\mathbb{R}^3 \times [0, T])$, $m > \frac{3}{2}$, is the unique local smooth solution of (4.15), then, by (4.25)–(4.26), there exist two vector fields $\mathbf{v}$ and $\mathbf{w}$, respectively, satisfying

$$(4.57) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \operatorname{div} \mathbf{v} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \mathbf{v}(\mathbf{x}, 0) = \phi_1(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

and

$$(4.58) \quad \begin{cases} \frac{\partial \mathbf{w}}{\partial t} - (\lambda + 2\mu) \Delta \mathbf{w} + (\mathbf{u} \cdot \nabla) \mathbf{w} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \operatorname{curl} \mathbf{w} = 0, & \mathbf{x} \in \mathbb{R}^3, 0 \leq t \leq T, \\ \mathbf{w}(\mathbf{x}, 0) = \phi_2(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

Conversely, If two vector fields $\mathbf{v}$ and $\mathbf{w}$ satisfy (4.58) and (4.57), respectively, then $\mathbf{v} + \mathbf{w} = \mathbf{u}$ uniquely satisfies the initial problem (4.18) (or equivalently, (4.15)) by employing $\mathbf{w} = \operatorname{grad} p$ (see, (4.23)).

4. We will show that

$$(4.59) \quad \|\mathbf{v}\|_{C(\mathbb{R}^3 \times [0, T])} \leq \|\phi_1\|_{C(\mathbb{R}^3)},$$

$$(4.60) \quad \|\mathbf{w}\|_{C(\mathbb{R}^3 \times [0, T])} \leq \|\phi_2\|_{C(\mathbb{R}^3)},$$

where $\mathbf{v}, \mathbf{w}, \phi_1, \phi_2$ are given in (4.57), (4.58) and (4.21). For arbitrary fixed $\epsilon > 0$, let $\tilde{\mathbf{v}}(\mathbf{x}, t) := e^{-\epsilon t} \mathbf{v}(\mathbf{x}, t)$, $\tilde{\mathbf{w}}(\mathbf{x}, t) := e^{-\epsilon t} \mathbf{w}(\mathbf{x}, t)$ and $\tilde{\mathbf{u}}(\mathbf{x}, t) := e^{-\epsilon t} \mathbf{u}(\mathbf{x}, t)$ for all $\mathbf{x} \in \mathbb{R}^3$. Note that $\mathbf{v}, \mathbf{w} \in C^1([0, T]; \mathbf{C}_0^2(\mathbb{R}^3)) \cap \mathbf{C}^\infty(\mathbb{R}^3 \times [0, T])$. Then, from (4.57) we get that $|\tilde{\mathbf{v}}|^2 = e^{-2\epsilon t}(v_1^2 + v_2^2 + v_3^2)$ satisfies the equation

$$(4.61) \quad \frac{\partial}{\partial t} |\tilde{\mathbf{v}}|^2 + 2\epsilon |\tilde{\mathbf{v}}|^2 + \mathbf{u} \cdot \nabla |\tilde{\mathbf{v}}|^2 - 2\mu \tilde{\mathbf{v}} \cdot \Delta \tilde{\mathbf{v}} = 0.$$

Since $2\tilde{\mathbf{v}} \cdot \Delta \tilde{\mathbf{v}} = \Delta |\tilde{\mathbf{v}}|^2 - 2|\nabla \tilde{\mathbf{v}}|^2$ we get

$$(4.62) \quad \frac{\partial}{\partial t} |\tilde{\mathbf{v}}|^2 + 2\epsilon |\tilde{\mathbf{v}}|^2 + \mathbf{u} \cdot \nabla |\tilde{\mathbf{v}}|^2 + \mu (-\Delta |\tilde{\mathbf{v}}|^2 + 2|\nabla \tilde{\mathbf{v}}|^2) = 0.$$

Therefore, if $|\tilde{\mathbf{v}}|^2$ has a local maximum at $(\mathbf{x}, t) \in \mathbb{R}^3 \times (0, T]$, then $\frac{\partial}{\partial t} |\tilde{\mathbf{v}}|^2 \geq 0$, $|\nabla \tilde{\mathbf{v}}|^2 = 0$ and $\Delta |\tilde{\mathbf{v}}|^2 \leq 0$ at this point $(\mathbf{x}, t)$. This implies that the left-hand side of (4.62) is positive unless $|\tilde{\mathbf{v}}|^2 = 0$. Hence

$$(4.63) \quad \|\mathbf{v}(t)\|_{L^\infty} \leq e^{\epsilon t} \|\mathbf{v}(0)\|_{L^\infty}.$$

Now (4.59) follows because $\epsilon > 0$ was arbitrary.

Similarly, since (4.58) has unique local smooth solution $\mathbf{w}$, it follows that $|\tilde{\mathbf{w}}|^2 = e^{-2\epsilon t}(w_1^2 + w_2^2 + w_3^2)$ satisfies the equation

$$(4.64) \quad \frac{\partial}{\partial t} |\tilde{\mathbf{w}}|^2 + 2\epsilon |\tilde{\mathbf{w}}|^2 + \mathbf{u} \cdot \nabla |\tilde{\mathbf{w}}|^2 + (\lambda + 2\mu) (-\Delta |\tilde{\mathbf{w}}|^2 + 2|\nabla \tilde{\mathbf{w}}|^2) = 0.$$

If $|\tilde{\mathbf{w}}|^2$ has a local maximum at $(\mathbf{x}, t) \in \mathbb{R}^3 \times (0, T]$ then $\frac{\partial}{\partial t} |\tilde{\mathbf{w}}|^2 \geq 0$, $|\nabla \tilde{\mathbf{w}}|^2 = 0$ and $\Delta |\tilde{\mathbf{w}}|^2 \leq 0$ at this point $(\mathbf{x}, t)$, which imply that the left-hand side of (4.62) is positive unless $|\tilde{\mathbf{w}}|^2 = 0$. Hence

$$\|\mathbf{w}(t)\|_{L^\infty} \leq e^{\epsilon t} \|\mathbf{w}(0)\|_{L^\infty},$$

which leads to

$$(4.65) \quad \|\mathbf{w}\|_{C(\mathbb{R}^3 \times [0, T])} \leq \|\phi_2\|_{C(\mathbb{R}^3)}.$$

Since the solution $\mathbf{u}$ of initial problem (4.15) is unique in $C([0, T]; H^{m+1}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)) \cap C^\infty(\mathbb{R}^3 \times [0, T])$, we see $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Combining (4.19), (4.22)–(4.57), (4.59), (4.65), we get the desired result (4.16). $\square$

Remark 4.4. *Theorem 4.3 only holds when $n = 3$ because we have used the Hodge decomposition of 3-dimensional vector field, in which the curl concept can be exactly defined for $n = 3$. However, our method is not suitable for $n \geq 4$.*

We remark that although (4.1) is a nonlinear parabolic system of equations, it is still a real analytic system. According to continuity method, the solution $\mathbf{u} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$ can be smoothly extended to a larger domain $\mathbb{R}^n \times [0, T]$ with $T > T_0$ such that $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n))$. We will give a criterion for the so-called “supremum of existence time” of the solution.

Lemma 4.5. *Let $n = 3$, and let $\phi \in \mathcal{S}(\mathbb{R}^n)$. If $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times [0, T])$, $m > \frac{n}{2}$, is a local smooth solution of (4.15) such that $T^* \in (0, \infty)$ is the supremum of all existence time T of the solution (i.e., no smooth solution exists on $[0, T^*]$), then $\sup_{[0, T^*)} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)} = \infty$.*

Proof. Suppose by contradiction that there exists smooth solution $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times [0, T])$ of (4.1) for each $T \in (0, T^*)$ and there does not exist smooth solution on $\mathbb{R}^n \times [0, T^*]$, but

$$(4.66) \quad \sup_{[0, T^*)} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)} < +\infty.$$

In view of Theorem 4.3, we get that there exists a positive constant C , which is independent of λ and T and depends only on μ , such that $\|\mathbf{u}\|_{C(\mathbb{R}^n \times [0, T])} \leq C\|\phi\|_{\mathbb{R}^n}$ for any $T \in (0, T^*)$. Therefore, from this and (4.66) we have $\sup_{T \in [T_0, T^*)} \|\mathbf{u}\|_{C(\mathbb{R}^n \times [0, T])} < +\infty$ and $\sup_{T \in [T_0, T^*)} \|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n))} < +\infty$. Let us define a new function $\mathbf{F}(\mathbf{x}, t) := -(\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\mathbf{u}(\mathbf{x}, t)$ for $\mathbf{x} \in \mathbb{R}^n$, $t \in [0, T^*]$ by giving some finite vectors on the boundary $\{T^*\} \times \mathbb{R}^n$. Clearly, $\mathbf{F} \in L^2([0, T^*]; \mathbf{L}^2(\mathbb{R}^n))$. It follows from the existence and uniqueness of weak solution by Galerkin approximation (see, Theorems 3, 4 on p.380-381 of [11]) there exists a weak solution $\mathbf{u}^* \in L^2([0, T^*]; \mathbf{H}^1(\mathbb{R}^n))$ of (4.1), with $\frac{\partial \mathbf{u}^*}{\partial t} \in L^2([0, T^*]; \mathbf{H}^{-1}(\mathbb{R}^n))$. Furthermore, since $\phi \in \mathcal{S}(\mathbb{R}^n)$, by regularity (see §7.1.3 of [11]) and by using bootstrap argument, we get $\mathbf{u}^* \in C([0, T^*]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times [0, T^*])$. This is a contradicts, which implies the desired result. $\square$

Theorem 4.6. *Let $n = 3$, and let $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times [0, T])$, $m > \frac{n}{2}$, satisfy*

$$(4.67) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

then

$$(4.68) \quad \frac{d}{dt} \|\mathbf{u}(t)\|_{\mathbf{H}^k(\mathbb{R}^n)}^2 \leq c_k \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)}^2 \|\mathbf{u}\|_{\mathbf{H}^k(\mathbb{R}^n)}^2, \quad \text{for all } k = 0, 1, 2, \dots, m$$

where the constant c_k is independent of λ , and depends only on $\mu > 0$ and k .

Proof. Noting that $\mathbf{H}^{m+1}(\mathbb{R}^n) = \mathbf{H}_0^{m+1}(\mathbb{R}^n)$, $m > \frac{n}{2}$, we find by compact imbedding theorem and trace theorem (see, Theorems 1 and 2, on p.274–275 of [11]) that $\lim_{|\mathbf{x}| \rightarrow +\infty} \partial^\alpha \mathbf{u}(\mathbf{x}) = 0$, $|\alpha| \leq 1$. First, by taking the inner product of $\mathbf{u}$ and equations (4.67) in $\mathbf{L}^2(\mathbb{R}^n)$, applying Green’s formula (see [33] or p.471 of [44]) and $\lambda + \mu \geq 0$, we find that

$$(4.69) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(\mathbf{x}, t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x}, t) \cdot \left(\mu \Delta \mathbf{u}(\mathbf{x}, t) + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u}(\mathbf{x}, t) - (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) d\mathbf{x}$$

$$\begin{aligned}
&= - \int_{\mathbb{R}^n} \left(2\mu(\text{Def } \mathbf{u}(\mathbf{x}, t), \text{Def } \mathbf{u}(\mathbf{x}, t)) + \lambda(\text{div } \mathbf{u}(\mathbf{x}, t))^2 \right. \\
&\quad \left. + \mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) dx \\
&= - \int_{\mathbb{R}^n} \left(\mu |\nabla \mathbf{u}(\mathbf{x}, t)|^2 + (\lambda + \mu)(\text{div } \mathbf{u}(\mathbf{x}, t))^2 \right. \\
&\quad \left. + \mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) dx \\
&\leq - \int_{\mathbb{R}^n} \mu |\nabla \mathbf{u}(\mathbf{x}, t)|^2 dx + \left| \int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) dx \right|,
\end{aligned}$$

where $\text{Def } \mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ and $(\text{Def } \mathbf{u}, \text{Def } \mathbf{u}) := \frac{1}{4} \sum_{j,k=1}^n \left| \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right|^2$. From Hölder's inequality, Theorem 4.3 and (4.2) we get

$$\begin{aligned}
(4.70) \quad &\left| \int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) dx \right| = \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n u_l(\mathbf{x}, t) u_j(\mathbf{x}, t) \frac{\partial u_l(\mathbf{x}, t)}{\partial x_j} dx \right| \\
&\leq \sum_{j,l=1}^n c_1 (\|u_l\|_{L^\infty(\mathbb{R}^n)} \|u_j\|_{L^2(\mathbb{R}^n)} + \|u_l\|_{L^2(\mathbb{R}^n)} \|u_j\|_{L^\infty(\mathbb{R}^n)}) \left\| \frac{\partial u_l}{\partial x_j} \right\|_{L^2(\mathbb{R}^n)} \\
&\leq c_2 \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)} \|\mathbf{u}\|_{L^2(\mathbb{R}^n)} \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^n)} \\
&\leq \frac{c_2^2}{2\mu} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}^2 \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + \frac{\mu}{2} \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where positive constant c_2 is independent of λ , and depends only on $\mu > 0$. It follows from (4.69) and (4.70) that

$$(4.71) \quad \frac{d}{dt} \|\mathbf{u}(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{c_2^2}{\mu} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}^2 \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2.$$

Next, since $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times [0, T])$ satisfies (4.67), we get $(\mathbf{u} \cdot \nabla) \mathbf{u} \in C([0, T]; \mathbf{H}^m(\mathbb{R}^n))$. Write (4.67) again as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \text{grad div } \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u}, & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

and applying regularity (cf. §7.1.3 of [11]) and bootstrap argument we get $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+2}(\mathbb{R}^n))$, so that by trace theorem (cf. p. 274-275 of [11]) we have

$$(4.72) \quad \lim_{|\mathbf{x}| \rightarrow +\infty} \Delta \mathbf{u}(\mathbf{x}, t) = 0 \quad \text{for each fixed } t > 0.$$

By taking the inner product of $\Delta \mathbf{u}$ and equation (4.67) in $L^2(\mathbb{R}^n)$, applying Green's formula and $\lambda + \mu \geq 0$, we get

$$\begin{aligned}
(4.73) \quad &\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}(\mathbf{x}, t)\|_{L^2(\mathbb{R}^n)}^2 = - \int_{\mathbb{R}^n} (\Delta \mathbf{u}(\mathbf{x}, t)) \cdot \left(\mu \Delta \mathbf{u}(\mathbf{x}, t) + (\lambda + \mu) \text{grad div } \mathbf{u}(\mathbf{x}, t) - (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) dx \\
&= - \int_{\mathbb{R}^n} \left(\mu (\Delta \mathbf{u}(\mathbf{x}, t))^2 + (\lambda + \mu) |\nabla(\text{div } \mathbf{u}(\mathbf{x}, t))|^2 \right. \\
&\quad \left. - (\Delta \mathbf{u}(\mathbf{x}, t)) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) dx \\
&\leq - \int_{\mathbb{R}^n} \mu (\Delta \mathbf{u}(\mathbf{x}, t))^2 dx + \left| \int_{\mathbb{R}^n} (\Delta \mathbf{u}(\mathbf{x}, t)) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) dx \right|.
\end{aligned}$$

Because of

$$\begin{aligned}
 (4.74) \quad \left| \int_{\mathbb{R}^n} (\Delta \mathbf{u}(\mathbf{x}, t)) \cdot (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) dx \right| &= \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\Delta u_l(\mathbf{x}, t)) u_j(\mathbf{x}, t) \frac{\partial u_l(\mathbf{x}, t)}{\partial x_j} dx \right| \\
 &\leq \sum_{j,l=1}^n \left(\|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \left\| \frac{\partial u_l}{\partial x_j} \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \right) \|\Delta u_l\|_{\mathbf{L}^2(\mathbb{R}^n)} \\
 &\leq c_3 \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)} \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)} \\
 &\leq \frac{c_3^2}{2\mu} \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)}^2 \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \frac{\mu}{2} \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)}^2,
 \end{aligned}$$

it follows from (4.73) and (4.74) that

$$(4.75) \quad \frac{d}{dt} \|\nabla \mathbf{u}(\mathbf{x}, t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \leq \frac{c_3^2}{\mu} \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)}^2 \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)}^2.$$

Consequently, (4.71) and (4.75) imply

$$(4.76) \quad \frac{d}{dt} \|\mathbf{u}(\mathbf{x}, t)\|_{\mathbf{H}^1(\mathbb{R}^n)}^2 \leq c_4 \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)}^2 \|\mathbf{u}\|_{\mathbf{H}^1(\mathbb{R}^n)}^2,$$

where constants c_1, c_2, c_3 and c_4 are independent of λ , and depend only on μ and n .

Now, taking the Laplacian on the both sides of (4.67) we have

$$(4.77) \quad \begin{cases} \frac{\partial(\Delta \mathbf{u})}{\partial t} - \mu \Delta^2 \mathbf{u} - (\lambda + \mu) \Delta \operatorname{grad} \operatorname{div} \mathbf{u} + \Delta((\mathbf{u} \cdot \nabla) \mathbf{u}) = 0, & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ \Delta \mathbf{u}(0, \mathbf{x}) = \Delta \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$, by taking the inner product of $\Delta \mathbf{u}$ and (4.77) in $L^2(\mathbb{R}^n)$ and applying Green's formula, we get the following inequality

$$\begin{aligned}
 (4.78) \quad \frac{1}{2} \frac{d}{dt} \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} (\Delta \mathbf{u}) \cdot \left(\mu \Delta^2 \mathbf{u} + (\lambda + \mu) \Delta \operatorname{grad} \operatorname{div} \mathbf{u} - \Delta((\mathbf{u} \cdot \nabla) \mathbf{u}) \right) dx \\
 &= - \int_{\mathbb{R}^n} \left(\mu |\nabla(\Delta \mathbf{u})|^2 + (\lambda + \mu) |\Delta \operatorname{div} \mathbf{u}|^2 \right) dx - \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\Delta u_l) \left[\Delta(u_j \frac{\partial u_l}{\partial x_j}) \right] dx \\
 &\leq - \int_{\mathbb{R}^n} \mu |\nabla(\Delta \mathbf{u})|^2 dx + \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\Delta u_l) \left[u_j \left(\Delta \frac{\partial u_l}{\partial x_j} \right) + (\Delta u_j) \frac{\partial u_l}{\partial x_j} + 2 \nabla u_j \cdot \nabla \left(\frac{\partial u_l}{\partial x_j} \right) \right] dx \right|.
 \end{aligned}$$

By Green's formula, we see that

$$\begin{aligned}
 (4.79) \quad &\left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\Delta u_l) \left[u_j \left(\Delta \frac{\partial u_l}{\partial x_j} \right) + (\Delta u_j) \frac{\partial u_l}{\partial x_j} + 2 \nabla u_j \cdot \nabla \left(\frac{\partial u_l}{\partial x_j} \right) \right] dx \right| \\
 &= \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n \left[-u_j (\Delta u_l) \left(\Delta \frac{\partial u_l}{\partial x_j} \right) - u_l (\Delta u_j) \left(\Delta \frac{\partial u_l}{\partial x_j} \right) - u_l (\Delta u_l) \left(\Delta \frac{\partial u_j}{\partial x_j} \right) \right] dx \right. \\
 &\quad \left. - 2 \int_{\mathbb{R}^n} \sum_{j,k,l=1}^n u_j \frac{\partial(\Delta u_l)}{\partial x_k} \frac{\partial^2 u_l}{\partial x_k \partial x_j} dx \right| \\
 &\leq c_5 \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \|\Delta \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)} \|\nabla(\Delta \mathbf{u})\|_{\mathbf{L}^2(\mathbb{R}^n)} + c_6 \|\mathbf{u}\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \|\nabla(\Delta \mathbf{u})\|_{\mathbf{L}^2(\mathbb{R}^n)} \|D^2 \mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^n)}.
 \end{aligned}$$

It is well-known (see, for example, (3) on p. 328 of [11], or (9.27) on p. 230 and p. 235 of [16]) that if $\Delta w \in L^2(\mathbb{R}^n)$ and $\lim_{|\mathbf{x}| \rightarrow +\infty} w(\mathbf{x}) = 0$, then there exists a positive constant c_7 depending only on n such that

$$(4.80) \quad \left(\int_{\mathbb{R}^n} \sum_{j,k=1}^n \left(\frac{\partial^2 w}{\partial x_j \partial x_k} \right)^2 dx \right)^{1/2} = \|D^2 w\|_{L^2(\mathbb{R}^n)} \leq c_7 \|\Delta w\|_{L^2(\mathbb{R}^n)}.$$

From this and (4.79) we obtain

$$(4.81) \quad \begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\Delta u_l) \left[u_j \left(\Delta \frac{\partial u_l}{\partial x_j} \right) + (\Delta u_j) \frac{\partial u_l}{\partial x_j} + 2 \nabla u_j \cdot \nabla \left(\frac{\partial u_l}{\partial x_j} \right) \right] dx \right| \\ & \leq c_8 \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)} \|\Delta \mathbf{u}\|_{L^2(\mathbb{R}^n)} \|\nabla(\Delta \mathbf{u})\|_{L^2(\mathbb{R}^n)} \\ & \leq \frac{c_8^2}{2\mu} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}^2 \|\Delta \mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + \frac{\mu}{2} \|\nabla(\Delta \mathbf{u})\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

so that by (4.81) and (4.78)

$$(4.82) \quad \frac{d}{dt} \|\Delta \mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{c_8^2}{\mu} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}^2 \|\Delta \mathbf{u}\|_{L^2(\mathbb{R}^n)}^2,$$

where positive constant c_8 is independent of λ , and depends only on $\mu > 0$.

Similarly, using bootstrap argument and regularity, taking the inner product of $\Delta^2 \mathbf{u}$ and equations (4.77), and applying Green's formula we get

$$(4.83) \quad \frac{d}{dt} \|\nabla(\Delta \mathbf{u})\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{c_9^2}{\mu} \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}^2 \|\nabla(\Delta \mathbf{u})\|_{L^2(\mathbb{R}^n)}^2.$$

Repeating this procedure again and again, we can get (4.68) for all $4 \leq k \leq m$. $\square$

Lemma 4.7. (i) (**Gronwall's inequality of differential form**). Let $\zeta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality

$$(4.84) \quad \zeta'(t) \leq \rho(t)\zeta(t) + \omega(t),$$

where $\rho(t)$ and $\omega(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$(4.85) \quad \zeta(t) \leq e^{\int_0^t \rho(s) ds} \left[\zeta(0) + \int_0^t \omega(s) ds \right] \quad \text{for all } 0 \leq t \leq T.$$

(ii) (**Gronwall's inequality of integral form**). Let χ, ζ, ρ be real-valued continuous functions for $0 \leq t \leq T$ and let $\rho(t) \geq 0$ for $0 \leq t \leq T$. If

$$(4.86) \quad \zeta(t) \leq \chi(t) + \int_0^t \rho(\tau) \zeta(\tau) d\tau \quad \text{for } 0 \leq t \leq T,$$

then

$$(4.87) \quad \zeta(t) \leq \chi(t) + \int_0^t \rho(\tau) \chi(\tau) \exp \left[\int_\tau^t \rho(\sigma) d\sigma \right] d\tau \quad \text{for a.e. } 0 \leq t \leq T.$$

Now, we can prove the main theorem of this section:

Theorem 4.8. Let $n = 3$, and let $\mu > 0$ and $\lambda + \mu \geq 0$. Given $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a global strong solution $\mathbf{u} \in C([0, \infty); \mathbf{H}^{m+1}(\mathbb{R}^n) \cap \mathbf{C}_0(\mathbb{R}^n))$, $m > n/2$, of the inertial parabolic Lamé equations (4.1). Moreover, this solution is unique and $\mathbf{u} \in \mathbf{C}^\infty(\mathbb{R}^n \times [0, \infty))$.

Proof. From Theorem 4.2, we know that for given $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a $T_0 > 0$ such that the initial problem (4.1) admits a unique smooth solution $\mathbf{u} \in C([0, T_0]; \mathbf{H}^{m+1}(\mathbb{R}^n))$. Recall that $T^* > 0$ is the supremum of all existence time T of the solution for the initial problem of the inertial Lamé equations. It follows that either $0 < T^* < \infty$ or $T^* = +\infty$. Suppose by contradiction that $0 < T^* < +\infty$. According to Lemma 4.5, we get

$$(4.88) \quad \sup_{[0, T^*)} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)} = \infty.$$

On the other hand, it follows from (4.16) of Theorem 4.3 that

$$(4.89) \quad \|\mathbf{u}(\mathbf{x}, t)\|_{C(\mathbb{R}^n \times [0, T])} \leq C \|\phi\|_{C(\mathbb{R}^n)}$$

for all $T < T^*$, where the constant C is independent of λ and T , and depends only on μ . By Theorem 4.6, we see that for any $t \in [0, T]$ with $T < T^*$,

$$(4.90) \quad \frac{d}{dt} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)}^2 \leq c_1 \|\mathbf{u}(t)\|_{C(\mathbb{R}^n)}^2 \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)}^2,$$

where the constant c_1 is independent of λ , and depends only on μ . Therefore, we find by Gronwall's inequality (4.85) and then by (4.89) that for any $t \in [0, T]$ with $T < T^*$,

$$(4.91) \quad \begin{aligned} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^n)}^2 &\leq \|\mathbf{u}(0)\|_{\mathbf{H}^1(\mathbb{R}^n)}^2 e^{\int_0^t c_1 \|\mathbf{u}(s)\|_{C(\mathbb{R}^n)}^2 ds} \\ &\leq \|\phi\|_{\mathbf{H}^1(\mathbb{R}^n)}^2 e^{t c_1 C \|\phi\|_{C(\mathbb{R}^n)}^2}. \end{aligned}$$

Note that $T^* c_1 C \|\phi\|_{C(\mathbb{R}^n)}^2 < +\infty$ since the positive constants C and c_1 are independent of λ , and depend only on μ . (4.89) and (4.91) rule out the blowup of $\mathbf{u}(t)$ in the $\mathbf{H}^1(\mathbb{R}^n)$ norm as $t \rightarrow T^*$, which contradicts (4.88). Thus, we must have $T^* = +\infty$, i.e., the solution can be smoothly extended over $[0, \infty)$, as required.

Now, we can show that the solution $\mathbf{u}$ of initial problem (4.1) is unique and smooth. It suffices to show that for any given $T > 0$, the solution $\mathbf{u} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n))$ of (4.1) is unique and smooth. Completely similar to the last part of proof of Theorem 4.2, if $\tilde{\mathbf{u}}$ is another solution of (4.1), then we get

$$\begin{cases} \frac{\partial(\mathbf{u} - \tilde{\mathbf{u}})}{\partial t} - \mu \Delta(\mathbf{u} - \tilde{\mathbf{u}}) - (\lambda + \mu) \operatorname{grad} \operatorname{div}(\mathbf{u} - \tilde{\mathbf{u}}) + (\mathbf{u} \cdot \nabla)(\mathbf{u} - \tilde{\mathbf{u}}) + ((\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla) \tilde{\mathbf{u}} = 0, & \mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T, \\ (\mathbf{u} - \tilde{\mathbf{u}})(0, \mathbf{x}) = 0, & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

which is a linear parabolic system with unknown vector-valued function $\mathbf{u} - \tilde{\mathbf{u}}$ and zero initial-value when we regard $\mathbf{u}$ and $\nabla \tilde{\mathbf{u}}$ as known coefficients. Thus, we have $\mathbf{u} \equiv \tilde{\mathbf{u}}$. Finally, we can bootstrap to show that solution is C^∞ -smooth in $\mathbb{R}^n \times [0, T]$ by standard regularity of parabolic equations (see, §7.1.3 of [11]). $\square$

Remark 4.9. *i) As pointed out before, when setting $\lambda + \mu = 0$, the inertial Lamé equations reduce to the viscous Burgers equations*

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

Thus, we have actually proved the existence, uniqueness and smoothness of the solution of the initial value problem for the viscous Burgers equations with $n = 3$ (cf. [18], [48], [37], or p. 324 of [45]).

ii) For the parabolic inertial Lamé equations, as in the case of linear parabolic Lamé equations, the initial $\phi \in \mathcal{S}(\mathbb{R}^3)$ will ensure any-order parabolic compatibility conditions to hold, so that the corresponding solutions can be C^∞ -smoothly extended to $\mathbb{R}^3 \times [0, \infty)$ (including C^∞ smoothly extended to the initial time $t = 0$) by using regularity and bootstrap argument.

iii) More importantly, the short-time solution $\mathbf{u}_{\lambda,\mu}(t)$ of the inertial Lamé equations can be uniquely obtained by an iterative sequence of the contraction map $\Psi \mathbf{w}(t) = e^{-tL_{\lambda,\mu}} \phi + \int_0^t e^{-(t-s)L_{\lambda,\mu}} \Phi(\mathbf{w}(s)) ds$ for given initial value ϕ . This provides a useful way to calculate the solution of (4.1) in the calculation mathematics and calculation physics.

iv) If the initial value $\phi \in \mathbf{H}^{m+1}(\mathbb{R}^3) \cap C_0^{m+1}(\mathbb{R}^3)$, ($m > \frac{3}{2}$), then from the above discussion, we can get a unique global smooth solution $\mathbf{u} \in C([0, \infty); \mathbf{H}^{m+1}(\mathbb{R}^3))$ for the inertial Lamé equations (4.1), with $\frac{\partial^k \mathbf{u}_{\lambda,\mu}}{\partial t^k} \in C([0, \infty); \mathbf{H}^{m+1-2k}(\mathbb{R}^3))$.

5. Existence and uniqueness of smooth solutions of the Navier-Stokes equations

From now on, we will assume $n = 3$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{div} \phi = 0$ in $\mathbb{R}^n$. As shown in the above section, for such an initial value ϕ , the inertial Lamé equations

$$(5.1) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

has a unique global smooth solution $\mathbf{u}(\mathbf{x}, t) \in C([0, \infty); \mathbf{H}^{m+1}(\mathbb{R}^n) \cap \mathbf{C}_0(\mathbb{R}^n)) \cap \mathbf{C}^\infty(\mathbb{R}^n \times [0, \infty))$, $m > n/2$. Since the Lamé constant $\mu > 0$ is fixed in advanced, for convenience, we will denote the unique global smooth solution of (5.1) as $\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)$ corresponding to each Lamé constant $\lambda \in [-\mu, \infty)$.

Proof of Theorem 1.1. We first consider the case of solutions in finite time interval $[0, T]$ for any fixed $T > 0$. It follows from imbedding theorem and trace theorem that $\mathbf{H}^{m+1}(\mathbb{R}^n) \subset C_0^1(\mathbb{R}^n)$ for $m > n/2$. We thus have that $\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t) \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap C([0, T]; C_0^1(\mathbb{R}^n))$, and that $\mathbf{u}_{\lambda,\mu}$ is also a smooth vector-valued function with respect to the parameter λ because the solutions of the parabolic system depend smoothly on the coefficients of the equations. From (4.16) of Theorem 4.3, we know that for any $t \in [0, T]$,

$$(5.2) \quad \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)\|_{\mathbf{C}(\mathbb{R}^n)} \leq C \|\phi\|_{\mathbf{C}(\mathbb{R}^n)},$$

where the constant C is independent of λ and T , and depends only on μ . According to (4.68) of Theorem 4.6 we know

$$(5.3) \quad \frac{d}{dt} \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)\|_{\mathbf{H}^k(\mathbb{R}^n)}^2 \leq c_k \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)\|_{\mathbf{C}(\mathbb{R}^n)}^2 \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)\|_{\mathbf{H}^k(\mathbb{R}^n)}^2, \quad \text{for all } k = 0, 1, 2, \dots, m,$$

so that by Gronwall's inequality and (5.2),

$$(5.4) \quad \begin{aligned} \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)\|_{\mathbf{H}^k(\mathbb{R}^n)}^2 &\leq \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, 0)\|_{\mathbf{H}^k(\mathbb{R}^n)}^2 e^{\int_0^t c_k \|\mathbf{u}_{\lambda,\mu}(\mathbf{x}, s)\|_{\mathbf{C}(\mathbb{R}^n)}^2 ds} \\ &\leq \|\phi\|_{\mathbf{H}^k(\mathbb{R}^n)}^2 e^{tc_k C \|\phi\|_{\mathbf{C}(\mathbb{R}^n)}^2} \quad \text{for all } k = 0, 1, 2, \dots, m \end{aligned}$$

where the constant $c_k C$ is independent of λ , and depends only on μ and k . By setting $\mathbf{F}_{\lambda,\mu}(\mathbf{x}, t) := -(\mathbf{u}_{\lambda,\mu}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}_{\lambda,\mu}(\mathbf{x}, t)$, we find from the above discussion that $\mathbf{F}_{\lambda,\mu} \in C([0, T]; \mathbf{H}^m(\mathbb{R}^n))$. Clearly,

$$(5.5) \quad \mathbf{F}_{\lambda,\mu} \in L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n))$$

and

$$(5.6) \quad \int_0^T \|\mathbf{F}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 dt = \int_0^T \int_{\mathbb{R}^n} |\mathbf{F}_{\lambda,\mu}(\mathbf{x}, t)|^2 dx dt \leq M,$$

where the constant M is independent of λ , and depends only on μ, T . Thus, our discussion becomes to first consider the initial problem of the following linear parabolic system (cf. [27] or §7.1.2 of [11]):

$$(5.7) \quad \begin{cases} \frac{\partial \mathbf{u}_{\lambda,\mu}}{\partial t} - \mu \Delta \mathbf{u}_{\lambda,\mu} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u}_{\lambda,\mu} = \mathbf{F}_{\lambda,\mu}(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^n, \quad 0 \leq t \leq T, \\ \mathbf{u}(0, \mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Since $\mathbf{H}^1(\mathbb{R}^n)$ is a separable Hilbert space, we can arbitrarily choose a basis $\{\mathbf{w}_j\}_{j=1}^\infty$ of $\mathbf{H}^1(\mathbb{R}^n)$. As obtained in the above section, $\mathbf{u}_{\lambda,\mu} \in C([0, T]; \mathbf{H}^{m+1}(\mathbb{R}^n)) \cap \mathbf{C}^\infty(\mathbb{R}^n \times [0, T])$ and $\lim_{|\mathbf{x}| \rightarrow \infty} \partial_{\mathbf{x}}^\alpha \mathbf{u}_{\lambda,\mu}(\mathbf{x}, t) = 0$ for all $|\alpha| \leq 1$ and $0 \leq t \leq T$. By taking the inner product of $\mathbf{u}_{\lambda,\mu}$ and (5.7) in $\mathbf{L}^2(\mathbb{R}^n)$ and employing Green's formula, we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \mu \|\nabla \mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 = \langle \mathbf{F}_{\lambda,\mu}(t), \mathbf{u}_{\lambda,\mu}(t) \rangle_{\mathbf{L}^2(\mathbb{R}^n)},$$

so that

$$(5.8) \quad \begin{aligned} \|\mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + 2 \int_0^t \left(\mu \|\nabla \mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \right) ds \\ \leq \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \|\mathbf{F}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds + \int_0^t \|\mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{L}^2(\mathbb{R}^n)}$ denotes the inner product in $\mathbf{L}^2(\mathbb{R}^n)$. Note that we have used the Hölder inequality:

$$\left| \langle \mathbf{F}_{\lambda,\mu}(t), \mathbf{u}_{\lambda,\mu}(t) \rangle_{\mathbf{L}^2(\mathbb{R}^n)} \right| \leq \frac{1}{2} \left(\|\mathbf{F}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \|\mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \right).$$

Hence

$$\|\mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \leq \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \|\mathbf{F}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds + \int_0^t \|\mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T,$$

or simply

$$(5.9) \quad \|\mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \leq C(\mu, \phi, T) + \int_0^t \|\mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T,$$

where $C(\mu, \phi, T) = \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + M$ denotes the constant which is independent of λ , and depends only on μ, ϕ and T , where M is the constant given in (5.6). Employing Gronwall's inequality to (5.9), we deduce that

$$(5.10) \quad \{\mathbf{u}_{\lambda,\mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^\infty([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

$$(5.11) \quad \{\mathbf{u}_{\lambda,\mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)).$$

Using (5.8) again, we get

$$2 \int_0^t \mu \|\nabla \mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds \leq \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \|\mathbf{F}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds + \int_0^t \|\mathbf{u}_{\lambda,\mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T,$$

which and (5.11) implies that

$$(5.12) \quad \{\nabla \mathbf{u}_{\lambda,\mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)).$$

By observing the system (5.7), we know $\frac{\partial \mathbf{u}_{\lambda,\mu}}{\partial t} \in C([0, T]; \mathbf{H}^{m-1}(\mathbb{R}^n))$. Taking the inner product of equations (5.7) with $\frac{\partial \mathbf{u}_{\lambda,\mu}(t)}{\partial t}$ in $\mathbf{L}^2(\mathbb{R}^n)$, we have

$$\left\| \frac{\partial \mathbf{u}_{\lambda,\mu}(t)}{\partial t} \right\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \frac{1}{2} \frac{d}{dt} \left(\mu \|\nabla \mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}_{\lambda,\mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \right) = \left\langle \mathbf{F}_{\lambda,\mu}(t), \frac{\partial \mathbf{u}_{\lambda,\mu}(t)}{\partial t} \right\rangle_{\mathbf{L}^2(\mathbb{R}^n)},$$

and integrating the result in $[0, T]$, obtaining

$$(5.13) \quad \begin{aligned} & \mu \|\nabla \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \left\| \frac{\partial \mathbf{u}_{\lambda, \mu}(s)}{\partial s} \right\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds \\ & \leq \mu \|\nabla \phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \|\mathbf{F}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T. \end{aligned}$$

Here we have used the condition $\operatorname{div} \mathbf{u}_{\lambda, \mu}(0) = \operatorname{div} \phi = 0$. Therefore

$$(5.14) \quad \mu \|\nabla \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \left\| \frac{\partial \mathbf{u}_{\lambda, \mu}(s)}{\partial s} \right\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds \leq \mu \|\nabla \phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + M, \quad 0 \leq t \leq T,$$

where M is the positive constant given in (5.6). It follows that

$$(5.15) \quad \{\nabla \mathbf{u}_{\lambda, \mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^\infty([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

$$(5.16) \quad \left\{ \frac{\partial \mathbf{u}_{\lambda, \mu}(t)}{\partial t} \right\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)).$$

As shown in (4.72), $\lim_{|\mathbf{x}| \rightarrow \infty} \Delta \mathbf{u}_{\lambda, \mu}(\mathbf{x}, t) = 0$ for each $t \in [0, T]$. Taking the inner product of $\Delta \mathbf{u}_{\lambda, \mu}$ and (5.7) in $\mathbf{L}^2(\mathbb{R}^n)$ and employing Green's formula, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \mu \|\Delta \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \nabla \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 = -\langle \mathbf{F}_{\lambda, \mu}(t), \Delta \mathbf{u}_{\lambda, \mu}(t) \rangle_{\mathbf{L}^2(\mathbb{R}^n)},$$

so that

$$\begin{aligned} & \|\nabla \mathbf{u}_{\lambda, \mu}(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + 2 \int_0^t \left(\mu \|\Delta \mathbf{u}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + (\lambda + \mu) \|\operatorname{div} \nabla \mathbf{u}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \right) ds \\ & \leq \|\nabla \phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \frac{1}{\mu} \|\mathbf{F}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds + \int_0^t \mu \|\Delta \mathbf{u}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T. \end{aligned}$$

This leads to

$$(5.17) \quad \int_0^t \mu \|\Delta \mathbf{u}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds \leq \|\nabla \phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \int_0^t \frac{1}{\mu} \|\mathbf{F}_{\lambda, \mu}(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds, \quad 0 \leq t \leq T,$$

i.e.,

$$(5.18) \quad \{\Delta \mathbf{u}_{\lambda, \mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)).$$

In addition, by (5.4) we actually get that for each fixed $k \geq 0$,

$$(5.19) \quad \{\mathbf{u}_{\lambda, \mu}(t)\}_{\lambda \in [-\mu, \infty)} \text{ is a bounded set of } L^2([0, T]; \mathbf{H}^k(\mathbb{R}^n)).$$

Next, we arbitrarily choose a sequence $\{\lambda_m\}_{m=1}^\infty$ such that $-\mu \leq \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots \rightarrow +\infty$. Then the conclusions (4.85), (5.10), (5.11), (5.12), (5.15), (5.16), (5.18) and (5.19) still hold when the set $\{\lambda \mid -\mu \leq \lambda < \infty\}$ is replaced by the sequence $\{\lambda_m\}_{m=1}^\infty$, since all our estimates above are independent of λ , and depend only on μ , ϕ and T . Note that $L^2([0, T]; \mathbf{H}^k(\mathbb{R}^n))$ is a Hilbert space for each non-negative interger k . Note also that $L^1([0, T]; \mathbf{L}^2(\mathbb{R}^n))$ is a separable Banach space, and its dual space is $L^\infty([0, T]; \mathbf{L}^2(\mathbb{R}^n))$. By compactness theorem, we find that there exists a subsequence $\{\lambda_{m_l}\}_{l=1}^\infty$ of $\{\lambda_m\}_{m=1}^\infty$ and a vector-valued function $\mathbf{u}_\mu \in L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n))$ such that as $l \rightarrow \infty$,

$$(5.20) \quad \mathbf{u}_{\lambda_{m_l}, \mu}(t) \xrightarrow{*} \mathbf{u}_\mu(t) \quad * \text{ weakly in } L^\infty([0, T]; \mathbf{H}^1(\mathbb{R}^n)),$$

$$(5.21) \quad \mathbf{u}_{\lambda_{m_l}, \mu}(t) \rightharpoonup \mathbf{u}_\mu(t) \quad \text{weakly in } L^2([0, T]; \mathbf{H}^1(\mathbb{R}^n)),$$

$$(5.22) \quad \frac{\partial \mathbf{u}_{\lambda_{m_l}, \mu}(t)}{\partial t} \rightharpoonup \frac{\partial \mathbf{u}_\mu(t)}{\partial t} \quad \text{weakly in } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

$$(5.23) \quad \nabla \mathbf{u}_{\lambda_{m_l}, \mu}(t) \rightharpoonup \nabla \mathbf{u}_\mu(t) \quad \text{weakly in } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

$$(5.24) \quad \Delta \mathbf{u}_{\lambda_{m_l}, \mu}(t) \rightharpoonup \Delta \mathbf{u}_\mu(t) \text{ weakly in } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

$$(5.25) \quad \mathbf{u}_{\lambda_{m_l}, \mu}(t) \rightharpoonup \mathbf{u}_\mu(t) \text{ weakly in } L^2([0, T]; \mathbf{H}^k(\mathbb{R}^n)).$$

We want to show that $\mathbf{u}_\mu$ is just the solution of the initial value problem for the Navier-Stokes equations. To this end, from (5.7), (5.20), (5.21), (5.22), (5.23), we see that for any $j \in \mathbb{N}$,

$$(5.26) \quad \left\langle \frac{\partial \mathbf{u}_{\lambda_{m_l}, \mu}(t)}{\partial t}, \mathbf{w}_j \right\rangle_{\mathbf{L}^2(\mathbb{R}^n)} + \mu \langle \nabla \mathbf{u}_{\lambda_{m_l}, \mu}(t), \nabla \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} \\ + \langle (\lambda_{m_l} + \mu) \operatorname{div} \mathbf{u}_{\lambda_{m_l}, \mu}, \operatorname{div} \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \langle \mathbf{F}_{\lambda_{m_l}, \mu}(t), \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)}$$

holds in $L^2[0, T]$. But, by (5.2), (5.5) and (5.20), (5.21), (5.22) and (5.23) we have

$$(5.27) \quad \lim_{l \rightarrow \infty} \left\langle \frac{\partial \mathbf{u}_{\lambda_{m_l}, \mu}(t)}{\partial t}, \mathbf{w}_j \right\rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \left\langle \frac{\partial \mathbf{u}_\mu(t)}{\partial t}, \mathbf{w}_j \right\rangle_{\mathbf{L}^2(\mathbb{R}^n)}, \\ \lim_{l \rightarrow \infty} \langle \nabla \mathbf{u}_{\lambda_{m_l}, \mu}(t), \nabla \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \langle \nabla \mathbf{u}_\mu(t), \nabla \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)}, \\ \lim_{l \rightarrow \infty} \langle \mathbf{F}_{\lambda_{m_l}, \mu}(t), \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \langle (\mathbf{u}_\mu(t) \cdot \nabla) \mathbf{u}_\mu(t), \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)}$$

holds in $L^2[0, T]$. Consequently, we find from this, (5.23) and (5.26) that $(\lambda_{m_l} + \mu) \operatorname{div} \mathbf{u}_{\lambda_{m_l}, \mu}$ must converge weakly in $\mathbf{L}^2(\mathbb{R}^n)$ as $l \rightarrow \infty$, saying, to a function $p_\mu(\mathbf{x}, t)$, and $p_\mu(\cdot, t) \in \mathbf{L}^2(\mathbb{R}^n)$ for all $0 \leq t \leq T$, i.e.,

$$(5.28) \quad \lim_{l \rightarrow \infty} \langle (\lambda_{m_l} + \mu) \operatorname{div} \mathbf{u}_{\lambda_{m_l}, \mu}, \operatorname{div} \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \langle p_\mu(t), \operatorname{div} \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)}$$

holds in $L^2[0, T]$. From (5.26), (5.27) and (5.28), we have

$$(5.29) \quad \left\langle \frac{\partial \mathbf{u}_\mu(t)}{\partial t}, \mathbf{w}_j \right\rangle_{\mathbf{L}^2(\mathbb{R}^n)} + \mu \langle \nabla \mathbf{u}_\mu(t), \nabla \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} \\ + \langle p_\mu(t), \operatorname{div} \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = \langle (\mathbf{u}_\mu(t) \cdot \nabla) \mathbf{u}_\mu(t), \mathbf{w}_j \rangle_{\mathbf{L}^2(\mathbb{R}^n)}$$

holds in $L^2[0, T]$. Clearly,

$$(5.30) \quad \operatorname{div} \mathbf{u}_{\lambda_{m_l}} \rightharpoonup \left(\lim_{l \rightarrow +\infty} \frac{1}{(\lambda_{m_l} + \mu)} \right) p_\mu = 0 \text{ weakly in } L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

i.e., $\operatorname{div} \mathbf{u}_{\lambda_{m_l}}$ weakly converges to 0 as $l \rightarrow +\infty$. Thus, $\operatorname{div} \mathbf{u}_\mu = 0$ in $\mathbb{R}^n \times [0, T]$ because the weak limit is unique. In view of $\mathbf{w}_j \in \mathbf{H}^1(\mathbb{R}^n)$, (5.29) can be written as

$$(5.31) \quad \left\langle \frac{\partial \mathbf{u}_\mu(t)}{\partial t} - \mu \Delta \mathbf{u}_\mu(t) - \operatorname{grad} p_\mu(\mathbf{x}, t), \mathbf{w}_j \right\rangle = -\langle (\mathbf{u}_\mu(t) \cdot \nabla) \mathbf{u}_\mu(t), \mathbf{w}_j \rangle$$

holds in $L^2[0, T]$, where $\langle \cdot, \cdot \rangle$ denotes the dual inner product of $\mathbf{H}^{-1}(\mathbb{R}^n)$ and $\mathbf{H}^1(\mathbb{R}^n)$. Because $\mathbf{w}_j$, ($j = 1, 2, \dots$), is a basis of $\mathbf{H}^1(\mathbb{R}^n)$, we find by (5.31) that

$$(5.32) \quad \frac{\partial \mathbf{u}_\mu(t)}{\partial t} - \mu \Delta \mathbf{u}_\mu(t) - \operatorname{grad} p_\mu(t) = -(\mathbf{u}_\mu(t) \cdot \nabla) \mathbf{u}_\mu(t),$$

i.e., $\mathbf{u}_\mu$ is a solution of the Navier-Stokes equations (5.32). Because $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\|\mathbf{u}_{\lambda_{m_l}, \mu}\|_{\mathbf{C}(\mathbb{R}^n)} \leq C\|\phi\|_{\mathbf{C}(\mathbb{R}^n)}$ (see, (5.2)) and $\mathbf{F}_{\lambda_{m_l}, \mu} := (\mathbf{u}_{\lambda_{m_l}, \mu} \cdot \nabla) \mathbf{u}_{\lambda_{m_l}, \mu} \in L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n))$, we see by (5.22) and (5.24) that

$$(5.33) \quad \frac{\partial \mathbf{u}_\mu(t)}{\partial t}, \mathbf{F}_\mu(t), \Delta \mathbf{u}_\mu(t) \in L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)),$$

so that by (5.32),

$$(5.34) \quad \mu \Delta \mathbf{u}_\mu + \operatorname{grad} p_\mu(t), \operatorname{grad} p_\mu(t) \in L^2([0, T]; \mathbf{L}^2(\mathbb{R}^n)).$$

Consequently, we obtain

$$(5.35) \quad \mathbf{u}_\mu \in L^2([0, T]; \mathbf{H}^2(\mathbb{R}^n)).$$

Now, we will show that $\mathbf{u}_\mu$ satisfies the initial-value conditions $\mathbf{u}_\mu(\mathbf{x}, 0) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. In fact, by virtue of $\mathbf{u}_{\lambda_{m_l}, \mu}(\mathbf{x}, 0) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $l \in \mathbb{N}$, it follows that

$$(5.36) \quad \mathbf{u}_{\lambda_{m_l}, \mu}(0) \rightharpoonup \mathbf{u}_\mu(0) \text{ weakly in } \mathbf{L}^2(\mathbb{R}^n) \text{ as } k \rightarrow \infty,$$

and hence $\mathbf{u}_\mu(0) = \phi$ by the uniqueness of the limit of a weak convergence sequence. Therefore, we have proved the existence of strong solution of the initial problem for the Navier-Stokes equations. More importantly, for another (arbitrarily chosen) sequence $-\mu \leq \lambda'_1 < \lambda'_2 < \dots < \lambda'_m < \dots \rightarrow +\infty$, we can show that the limit (vector-valued function) of $\mathbf{u}_{\lambda'_{m_l}, \mu}$ still is the same solution $\mathbf{u}_\mu$. In other words, the solution $\mathbf{u}_\mu$ is independent of the given sequence $\{\lambda_m\}$ (see also below for the proof of uniquenesses of the solution). Applying the standard regularity technique with respect to $\mathbf{x}$ -derivatives and t -derivatives, with the help of (5.25) (which implies $\mathbf{u}_\mu \in L^2([0, T]; \mathbf{H}^k(\mathbb{R}^n))$ for each non-negative integer k), we can further show that $\mathbf{u}_\mu \in \mathbf{C}^\infty(\mathbb{R}^n \times [0, T])$. We will omit the proof of this part (cf. [47], or §7.1.3 of [11]).

In the above discussion, since $T > 0$ can be taken as an arbitrary real number, we obtain that $(\mathbf{u}_\mu, p_\mu)$ is a smooth solution of initial-value problem for the following Navier-Stokes equations:

$$(5.37) \quad \begin{cases} \frac{\partial \mathbf{u}_\mu}{\partial t} - \mu \Delta \mathbf{u}_\mu + \text{grad } p_\mu + (\mathbf{u}_\mu \cdot \nabla) \mathbf{u}_\mu = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \text{div } \mathbf{u}_\mu = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \mathbf{u}_\mu(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Furthermore, the solution $(\mathbf{u}_\mu, p_\mu)$ of (5.37) is unique. In fact, if $(\tilde{\mathbf{u}}, \tilde{p})$ is another solution of (5.37), that is,

$$(5.38) \quad \begin{cases} \frac{\partial \tilde{\mathbf{u}}}{\partial t} - \mu \Delta \tilde{\mathbf{u}} + \text{grad } \tilde{p} + (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \text{div } \tilde{\mathbf{u}} = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \tilde{\mathbf{u}}(\mathbf{x}, 0) = \phi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

then

$$(5.39) \quad \begin{cases} \frac{\partial(\mathbf{u}_\mu - \tilde{\mathbf{u}})}{\partial t} - \mu \Delta(\mathbf{u}_\mu - \tilde{\mathbf{u}}) + \text{grad}(p_\mu - \tilde{p}) + ((\mathbf{u}_\mu - \tilde{\mathbf{u}}) \cdot \nabla) \mathbf{u}_\mu + (\tilde{\mathbf{u}} \cdot \nabla)(\mathbf{u}_\mu - \tilde{\mathbf{u}}) = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ \text{div}(\mathbf{u}_\mu - \tilde{\mathbf{u}}) = 0, & \mathbf{x} \in \mathbb{R}^n, t \geq 0, \\ (\mathbf{u}_\mu - \tilde{\mathbf{u}})(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

Thus,

$$(5.40) \quad \begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 - \mu \langle \Delta(\mathbf{u}_\mu - \tilde{\mathbf{u}}), (\mathbf{u}_\mu - \tilde{\mathbf{u}}) \rangle_{L^2(\mathbb{R}^n)} + \langle \text{grad}(p_\mu - \tilde{p}), (\mathbf{u}_\mu - \tilde{\mathbf{u}}) \rangle_{L^2(\mathbb{R}^n)} \\ + \langle ((\mathbf{u}_\mu - \tilde{\mathbf{u}}) \cdot \nabla) \mathbf{u}_\mu, (\mathbf{u}_\mu - \tilde{\mathbf{u}}) \rangle_{L^2(\mathbb{R}^n)} + \langle (\tilde{\mathbf{u}} \cdot \nabla)(\mathbf{u}_\mu - \tilde{\mathbf{u}}), (\mathbf{u}_\mu - \tilde{\mathbf{u}}) \rangle_{L^2(\mathbb{R}^n)} = 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 - \mu \langle \Delta(\mathbf{u}_\mu - \tilde{\mathbf{u}}), (\mathbf{u}_\mu - \tilde{\mathbf{u}}) \rangle_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} (\text{grad}(p_\mu - \tilde{p})) \cdot (\mathbf{u}_\mu - \tilde{\mathbf{u}}) dx \\ + \int_{\mathbb{R}^n} \sum_{j,l=1}^n ((\mathbf{u}_\mu)_j - \tilde{u}_j) \frac{\partial((\mathbf{u}_\mu)_l)}{\partial x_j} ((\mathbf{u}_\mu)_l - \tilde{u}_l) dx + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^n \tilde{u}_j \left(\frac{\partial}{\partial x_j} \sum_{l=1}^n ((\mathbf{u}_\mu)_l - \tilde{u}_l)^2 \right) dx = 0. \end{aligned}$$

Applying Green's formula, we get

$$\frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 + \mu \|\nabla(\mathbf{u}_\mu - \tilde{\mathbf{u}})\|_{L^2(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} (p_\mu - \tilde{p}) \text{div}(\mathbf{u}_\mu - \tilde{\mathbf{u}}) dx$$

$$- \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\mathbf{u}_\mu)_l \frac{\partial}{\partial x_j} [((\mathbf{u}_\mu)_j - \tilde{u}_j)((\mathbf{u}_\mu)_l - \tilde{u}_l)] dx - \frac{1}{2} \int_{\mathbb{R}^n} \sum_{l=1}^n ((\mathbf{u}_\mu)_l - \tilde{u}_l)^2 \operatorname{div} \tilde{\mathbf{u}} dx = 0,$$

and hence, by the conditions $\operatorname{div} \mathbf{u}_\mu = 0$ and $\operatorname{div} \tilde{\mathbf{u}} = 0$,

$$(5.41) \quad \frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 + \mu \|\nabla(\mathbf{u}_\mu - \tilde{\mathbf{u}})\|_{L^2(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\mathbf{u}_\mu)_l ((\mathbf{u}_\mu)_j - \tilde{u}_j) \frac{\partial((\mathbf{u}_\mu)_l - \tilde{u}_l)}{\partial x_j} dx = 0.$$

From Theorem 4.3 and (5.41), we have

$$(5.42) \quad \begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 + \mu \|\nabla(\mathbf{u}_\mu - \tilde{\mathbf{u}})\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \left| \int_{\mathbb{R}^n} \sum_{j,l=1}^n (\mathbf{u}_\mu)_l ((\mathbf{u}_\mu)_j - \tilde{u}_j) \frac{\partial((\mathbf{u}_\mu)_l - \tilde{u}_l)}{\partial x_j} dx \right| \\ & \leq \|\mathbf{u}_\mu\|_{C(\mathbb{R}^n \times [0, \infty))} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)} \|\nabla(\mathbf{u}_\mu - \tilde{\mathbf{u}})\|_{L^2(\mathbb{R}^n)} \\ & \leq \frac{1}{4\mu} \|\mathbf{u}_\mu\|_{C(\mathbb{R}^n \times [0, \infty))}^2 \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 + \mu \|\nabla(\mathbf{u}_\mu - \tilde{\mathbf{u}})\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

so that

$$(5.43) \quad \frac{\partial}{\partial t} \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{1}{2\mu} \|\mathbf{u}_\mu\|_{C(\mathbb{R}^n \times [0, \infty))}^2 \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2.$$

It follows from Gronwall's inequality (see, (i) of Lemma 4.7) that

$$(5.44) \quad \|\mathbf{u}_\mu - \tilde{\mathbf{u}}\|_{L^2(\mathbb{R}^n)}^2 \leq \left(e^{\int_0^t \frac{1}{2\mu} \|\mathbf{u}_\mu\|_{C(\mathbb{R}^n \times [0, t])}^2 ds} \right) \|\mathbf{u}_\mu(0) - \tilde{\mathbf{u}}(0)\|_{L^2(\mathbb{R}^n)}^2 = 0, \quad \text{for all } t \geq 0,$$

which implies $\mathbf{u}_\mu \equiv \tilde{\mathbf{u}}$.

Finally, taking the inner product of (5.37) and $\mathbf{u}_\mu$ in $\mathbf{L}^2(\mathbb{R}^n)$, we find by $\operatorname{div} \mathbf{u}_\mu = 0$ in $\mathbb{R}^n \times [0, \infty)$ that

$$(5.45) \quad \frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_\mu\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + \mu \|\nabla \mathbf{u}_\mu\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 = 0 \quad \text{for all } 0 \leq t < \infty,$$

so that

$$(5.46) \quad \|\mathbf{u}_\mu(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 + 2\mu \int_0^t \|\nabla \mathbf{u}_\mu(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds = \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2, \quad \text{for } 0 \leq t < \infty.$$

To obtain the above (5.45), we have used the fact that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbf{u}_\mu \cdot \operatorname{grad} p_\mu dx &= - \int_{\mathbb{R}^n} (\operatorname{div} \mathbf{u}_\mu) p_\mu dx = 0, \\ \int_{\mathbb{R}^n} \mathbf{u}_\mu \cdot (\mathbf{u}_\mu \cdot \nabla) \mathbf{u}_\mu dx &= \int_{\mathbb{R}^n} \mathbf{u}_\mu \cdot \nabla \left(\frac{1}{2} |\mathbf{u}_\mu|^2 \right) dx \\ &= - \int_{\mathbb{R}^n} \frac{1}{2} |\mathbf{u}_\mu|^2 \operatorname{div} \mathbf{u}_\mu dx = 0. \end{aligned}$$

From (5.46) we immediately get

$$(5.47) \quad \|\mathbf{u}_\mu(t)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \leq \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 \quad \text{for all } 0 \leq t < \infty$$

and

$$(5.48) \quad 2\mu \int_0^t \|\nabla \mathbf{u}_\mu(s)\|_{\mathbf{L}^2(\mathbb{R}^n)}^2 ds \leq \|\phi\|_{\mathbf{L}^2(\mathbb{R}^n)}^2, \quad \text{for } 0 \leq t < \infty,$$

i.e., the energy of solution $\mathbf{u}_\mu$ is bounded. More precisely, the solution $\mathbf{u}_\mu$ also satisfies the requirement of (1.7). $\square$

Remark 5.1. *i) We have proved the existence and uniqueness of global smooth solution for the Navier-Stokes equations in dimension $n = 3$ because Theorem 4.3 only holds when dimension $n = 3$.*

ii) The advantage of our method is that the existence and uniqueness of global smooth solution of the Navier-Stokes equations are completely proved by discussing the closed relationship of the nonlinear elastic inertial parabolic equations and the Navier-Stokes equations, which has avoided to work on the Leray projection and its associated estimates.

iii) Our method (for proving the existence of global smooth solution) will provide a very useful approach (in computational mathematics and computation physics) to calculate the solution $\mathbf{u}_\mu$ of the Navier-Stokes equations by first figuring out the solution $\mathbf{u}_{\lambda,\mu}$ of the inertial Lamé equations and then letting parameter $\lambda \rightarrow +\infty$ ($\mu > 0$ is fixed).

iv) As pointed out before, the conditions $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{div} \phi = 0$ in $\mathbb{R}^3$ will ensure that the solution of the Navier-Stokes equations can be C^∞ -smoothly extended to $\mathbb{R}^3 \times [0, \infty)$ (including smoothly extended to the initial time $t = 0$), just as predicted by C. L. Fefferman in [12]. However, if the initial value $\phi \in \mathbf{H}^{m+1}(\mathbb{R}^3) \cap \mathbf{C}_0^{m+1}(\mathbb{R}^3)$, ($m > \frac{3}{2}$), then from the previous discussion, we see that the Navier-Stokes equations (5.1) admits a unique global smooth solution $\mathbf{u}_\mu \in C([0, \infty); \mathbf{H}^{m+1}(\mathbb{R}^3)) \cap L^2([0, \infty); \mathbf{H}^1(\mathbb{R}^3))$, with $\frac{\partial^k \mathbf{u}_\mu}{\partial t^k} \in C([0, \infty); \mathbf{H}^{m+1-2k}(\mathbb{R}^3))$.

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